ELEMENTS

OF

ALGEBRA.

BY

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CHARLES PHILLIPS,
Prof. Civil Engineering, University, N. C.

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ELEMENTS OF ALGEBRA.

Article 1.—A mathematical principle is a truth admitted as self-evident, or proved by a course of reasoning called a demonstration. Thus, it is a self-evident principle, that if equals be added to equals, the results will be equal. But it requires a demonstration to show that if the sum of two quantities be added to the difference of those quantities, the result will be equal to twice the greater quantity.

2. Science is knowledge gained by theory or experiment, employed in the investigation of principles. Art is the application of acquired principles to the practical purposes of life.

3. Quantity is anything that can be measured or numbered. A thing is said to be *measured* when its magnitude is expressed in terms of the unit of measure, and is said to be *numbered* when this unit is unknown or indefinite.

Thus, 12 bushels of corn is a definite measure, the unit of measure being one bushel of corn. But a quantity, expressed by the number 12, or even by 12 bushels, would only be numbered, for it might be 12 bushels of gold, or 12 of flour, or 12 of anything else.

Lines are quantities, because they can be measured in terms of yards, feet, inches, &c. Time is quantity, being measured by hours, minutes, seconds, &c. The unit of measure for time is generally the second.

The operations of the mind, such as hope, fear, joy, grief, &c., are not quantities. For, although we speak of a great hope and a small hope, there is no definite unit of comparison by which to measure its magnitude.

Numbers are not quantities, but simply the agents by which to express the abstract relations between quantities of the same kind; that is to say, every number may be regarded as the quotient arising from dividing one quantity by another of the same kind, independently of the species to which they belong.

Thus, the number 12 may express the abstract relation between 12 inches and 1 inch, or between 12 months and 1 month; and may, therefore, represent the quotient arising from the division of the foot by the inch, or the year by the month. And so it may express the quotient between any two quantities whatever, provided they are of the same kind.

4. Mathematics is the science of quantity.

Arithmetic is that branch of mathematics in which the quantities considered are represented by numbers.

5. The word Algebra is of Arabic origin, and signifies to consolidate.

Algebra enables us to investigate arithmetical principles in a consolidated, and, at the same time, general manner, and may be regarded as a compendious, and also universal arithmetic.

6. The quantities considered in Algebra are represented by numbers and letters, and the operations to be performed are indicated by signs. The numbers, letters, and signs are generally called *symbols.*

The numbers and letters that represent quantities are, for convenience, most usually called quantities themselves. The student, however, should remember that they are only the representatives of quantities.

7. The sign $+$ is called *plus,* i. e., *more,* and when prefixed to a quantity, signifies that it is to be added to some other quantity expressed or understood. Thus, $a + b$ is read a plus b, and signifies that b is to be added to a. The expression $+ c$ signifies that c is to be added to some quantity not expressed. When no sign is written before a quantity, the sign $+$ is always understood. In the expression $a + b$, a is understood to be affected with the plus sign.

8. A horizontal line, thus, — is called *minus,* i. e. *less,* and when prefixed to a quantity, signifies that it is to be substracted from some other quantity, expressed or understood.

Thus, $a - b$ is read a minus b, and signifies that b is to be taken from a. The expression $-c$ signifies that c is to be taken from some quantity not expressed.

9. A Greek cross $\times$ is called the sign of multiplication, and when placed between quantities, indicates that they are to be multiplied together. Thus, $a \times b$ is read a multiplied by b, or simply a into b, the sign indicating a multiplication to be performed.

10 The multiplication of quantities is sometimes indicated by a point. Thus, $a \, . \, b$ indicates that a and b are to be multiplied together.

11. The multiplication of literal factors is usually indicated by writing them one after another, thus, $a\,b\,c$ is the same as $a \times b \times c$. This notation cannot be employed when numbers are used, for the product thus expressed would be confounded with some other number. The multiplication of 2 by 4, for instance, cannot be indicated by writing the one after the other, because the product would be mistaken for 42 or 24.

12. When several terms connected by the sign +, or —, are to be multiplied by a single term, the multiplication is indicated by means of parentheses. Thus, $(a + b + c)\, m$, signifies that the sum of a, b, and c is to be multiplied by m. When the multiplier itself is composed of more than one term, it is also enclosed in parenthesis. Thus, $(a + b)$ $(m - c)$ indicates that the sum of a and b is to be multiplied by the difference of m and c. A horizontal or vertical line is also used to collect terms for multiplication. Thus, $a \times \overline{m + n + c}$, indicates that the sum of m, n, and c is to be multiplied by a. The same thing may be indicated by a vertical bar, thus

$$\left.\begin{array}{r} +m \\ +n \\ +c \end{array}\right| a.$$

13. The coefficient of a quantity is a number or letter prefixed to a quantity, showing how often it has been added to itself. Thus, instead of writing $a + a + a$, which denotes the addition of a three times, we abridge the notation by writing $3a$. So also, $10xy$, signifies the addition of xy ten times. In like manner, mx signifies the addition of x, m times. The coefficient serves as a brief mode of indicating the addition of a quantity to itself. When no coefficient is written, 1 is always to be understood.

14. The exponent is a small number or letter written a little above and to the right of a number or letter, and indicates the number of times it enters into itself as a factor. Thus, we write

a^2 instead of aa, and call the result a square.
a^3 " " aaa, " " " " a cube.
a^4 " " $aaaa$, " " " " a to the fourth power.

The exponent enables us to abbreviate the manner of indicating the multiplication of a quantity by itself. When no exponent is written, 1 is always to be understood.

Division is denoted by three signs. The division of a by b may be indicated by $a \div b$, or $\frac{a}{b}$, or $a|\underline{b}$.

15. The sign $=$ is called the sign of equality, and is read "equal to." When placed between two quantities, it indicates that they are equal to each other. Thus, $a = b$, is read a equal to b. In like manner, $2 + 4 = 6$ is read 2 plus 4 equal to 6.

16. The sign $>$ is called the sign of inequality, and is read "greater than," when the opening is toward the left; and "less than," when opening is toward the right. Thus, $a > b$ is read, a greater than b; and $c < m$ is read, c less than m.

17. A root of a quantity is a quantity which, multiplied by itself a certain number of times, will produce the given quantity. To indicate the extraction of a root, we use the sign $\sqrt{\ }$, called the radical sign, placing a number or letter to the left and over the sign to indicate what root is to be extracted. Thus, $\sqrt[2]{a}$ denotes the square root of a; $\sqrt[3]{a}$ the cube root of a; $\sqrt[n]{a}$ the n^{th} root of a.

18. The number or letter placed over the radical sign is called the index of the radical. When no index is written, we always understand that the square root is to be extracted. Thus, $\sqrt{a}$ means that the square root of a is to be taken.

19. A simple quantity is one in which the letters and numbers of which it is composed are not connected by the sign plus or minus. Thus, a, ab, $\frac{a}{b}$ and c are simple quantities or monomials. All quantities not simple are compound, and called binomials when composed of two terms; trinomials when composed of three; and polynomials when composed of more than three. Each of the literal factors which enter into a term is called a dimension of the term. The degree of a term is the number of its dimensions. When a factor enters more than once, its dimension is denoted by the exponent. Thus, a^2 is of the second dimension.

20. When several factors are multiplied together, the sum of the exponents denotes the dimension of the term. Thus, ab is of the second dimension, a^2bc, of the fourth, ab^3cd, of the sixth, &c. If the term is a fraction, its degree is denoted by the difference between

the sums of the exponents of the numerator and denominator. Thus $\frac{a^2b}{c}$ is of the second degree, $\frac{a^2b^2}{c^3}$ of the first degree, &c.

21. A monomial is said to be *homogeneous*, when all of its literal factors are of the same dimension. A polynomial is said to be homogeneous, when all its terms are of the same degree. Thus ab, a^2b^2, a^3b^3, are each separately homogeneous monomials; $4a^2b+2b^2c-m^3$, is a homogeneous polynomial.

22. *Like quantities* are those which are composed of the same letters, and which have their corresponding letters affected with the same exponents; $3ab^2+10ab^2-3ab^2$, are like quantities. But $3ab^2+10ab-3a^2b^2$ are unlike; for, though the letters are like, the exponents of these letters are different. p. 17.

23. The *reciprocal* of a quantity is 1 divided by that quantity. The reciprocal of 2 is $\frac{1}{2}$; of a, $\frac{1}{a}$; of $\frac{2}{3}$. $\frac{1}{\frac{2}{3}}$ or $\frac{3}{2}$, &c.

24. Quantities, affected with the plus sign, are called *positive quantities*, and those affected with the minus sign, are called *negative quantities*. The student, however, should bear in mind that quantities cannot be positive or negative in themselves, and that by these phrases, we wish merely to signify additive and subtractive quantities.

25. To familiarize the student with the foregoing symbols, we subjoin a few examples for practice:—

1. Express in algebraic language that twice the product of x into y, divided by three times the product of z into w, shall be equal to 6.

2. Express that the sum of a and b, when added to their difference is equal to twice the greater quantity a.

3. That one-third of a quantity m, multiplied by $\frac{1}{5}$th of a second quantity n, and that product increased by 100, the result will be a hundred thousand.

4. Find the value of this expression, $\frac{b+\sqrt{c}}{m.n}$, when $b=64$, $c=144$, $m=10$, $n=1$.

Find the value of the expression $\frac{m^2-n^2}{m+n}(x\,a)$, when $m=4$, $n=3$, $x=2$, $a=1$.

Find the value of the expression $\frac{h^2cl}{m^2n}$, when $m=4$, $n=3$, $c=5$, $h=10$, $l=100$.

ADDITION.

26. Addition is the connecting of several terms together by the sign *plus* or *minus*, so that they may be reduced into a single expression.

There are three distinct cases: —

CASE I.

27. When the terms are like and have like signs.

RULE.

Add the coefficients of the several terms together, prefix the common sign to this sum, and write after it the common letter or letters, with their primitive exponents.

Thus, $+2a+3a$ are like quantities referred to the same unit, and may therefore be added, just as two whole numbers are added in Arithmetic. The $+2a$ indicates that a is to be added twice to some quantity not expressed, and $+3a$ that a is again to be added three times to the same quantity. This addition can obviously be indicated at once by writing $+5a$ instead of $+2a+3a$. Suppose, for example, that a represented one dollar, then $2a$ would represent two dollars, and $3a$ three dollars, and the sum, five dollars, would, of course, be represented by $5a$. In like manner, $-3b-4b$ is equal to $-7b$, because the minus signs indicate that the quantities represented by $3b$ and $4b$ are to be taken from some quantity not expressed, and subtracting $7b$ at once is obviously the same as first subtracting $3b$ and then subtracting $4b$. We may then write $-3b-4b=-7b$, the minus sign before the $7b$ indicating that it is to be taken from some quantity not expressed, and the whole expression denoting that the subtraction of $7b$ is the same as the successive subtraction of $3b$ and $4b$. This will be made plainer to the beginner by attributing to b some known value, as a pound or an ounce.

EXAMPLES.

$+a$	$-a$	$+b-y$	$+m-z$	$-4z^2$	$+2y^3$
$+2a$	$-2a$	$+2b-2y$	$+5m-2z$	$-5z^2$	$+5y^3$
$+3a$	$-3a$	$+5b-4y$	$+8m-3z$	$-9z^2$	$+7y^3$
$+4a$	$-4a$	$+9b-8y$	$+10m-7z$	$-12z^2$	$+8y^3$
$+10a$	$-10a$	$+17b-15y$	$+24m-13z$	$-30z^2$	$+22y^3$

CASE II.

28. When the quantities are like, but affected with unlike signs.

RULE.

Add the quantities affected with the positive sign by the last rule, then add those affected with the negative sign in like manner. Subtract the smaller of the coefficients of those sums from the greater, annex the common letter or letters to the difference, and prefix the sign of the greater sum.

If we were required to add $+5a - 6a + 4a - a$ together, we could, for the reasons already given, write $+9a$ for the positive terms, and $7a$ for the negative terms. By this, we must understand that $9a$ has to be added to some quantity not expressed, and that $7a$ has to be subtracted from the sum of $9a$, and this unexpressed quantity. Now to take $7a$ from $9a$, and add the remainder to the unexpressed quantity is plainly the same as taking $7a$ from the sum of $9a$, and the unexpressed quantity. But the difference between $9a$ and $7a$ is $2a$, hence the sum of $+5a - 6a + 4a - a$ is $+2a$. The plus $2a$ denotes that $2a$ has to be added to an unexpressed quantity.

If we were required to add $-5a + 6a - 4a + a$ together, we could, as has been shown, collect the negative terms into a single term, $-9a$, and the positive terms into a single term $+7a$. We would then be required to take $9a$ from the sum of $7a$ and some unexpressed quantity, or, which would be the same thing, to take $9a$ from $7a$, and add the remainder to the unexpressed quantity. But $9a$ is made up of $7a$ added to $2a$, and when, therefore, we subtract $9a$ from $7a$, the $7a$'s will strike each other out, and there will still remain $2a$ to be subtracted. The required and unexecuted subtraction is indicated by the minus sign before the $2a$, and the true sum of $-5a + 6a - 4a + a$ is therefore $-2a$.

EXAMPLES.

1. Add $2\,b - 6b + ab - 2b$ together.

   Ans. $+ab - 6b$ or $(+a - 6)\,b$.

2. Add $4c + 3c - mc + nc$ together.

   Ans. $(7 + n - m)\,c$.

3. Add $xy + 2xy + bxy - pxy$ together.

   Ans. $(3 + b - p)\,xy$.

4. Add $4a - 7a - 3a - 10a - 12a$ together.

Ans. $-28a$.

5. Add $4ax - 7ax - 3ax - 10ax - 12ax$ together.

Ans. $-28ax$.

CASE III.

29. When the quantities are similar and dissimilar, and have like and unlike signs.

RULE.

Add all the sets of similar terms by the last two rules, and write their sums one after another, and connect with them all the single terms with their own signs.

Since the letters are always the representatives of quantities, it is obvious that the quantities represented by dissimilar terms cannot be added into one sum. Thus, $4a + 4b$ neither make $8a$ nor $8b$. The quantity a might represent a year, and the quantity b a pound; then $4a$ would represent 4 years, and $4b$ would represent 4 pounds, and the addition ought neither to give 8 years nor 8 pounds. Hence, we can only reduce the similar terms in sets, and connect the results with their appropriate signs.

EXAMPLES.

$3m^2 \quad -n^2$	$xy \quad +7x$	$5ax \quad +y$	$2nx+w$
$3x^2 \quad +m^2$	$z \quad +y$	$x^2 \quad +4y$	$2w+5nx$
$2m^2 \quad +x^2$	$xy \quad +3x$	$4x^2 \quad +9y$	$6nx+s$
$6m^2+4x^2-n^2$	$2xy+10x+z+y$	$5x^2+5ax+14y$	$13nx+3w+s$

30. It often facilitates an algebraic operation to arrange the terms in a certain order. In addition, this arrangement is effected by placing all the like terms beneath one another.

$$\begin{array}{l} 7a^2+6xy+5z+4w \\ w+4z+8xy+12a^2 \\ 3z+9xy+5w+100a^2 \end{array} \quad \text{may be written} \quad \begin{array}{rrrr} 7a^2+ & 4w+ & 6xy+ & 5z \\ 12a^2+ & w+ & 8xy+ & 4z \\ 100a^2+ & 5w+ & 9xy+ & 3z \\ \hline 119a^2+ & 10w+ & 23xy+ & 12z \end{array}$$

31. Sometimes the addition of a compound quantity to a single quantity, or to another compound quantity, is not actually performed, but indicated by the parenthesis ( ), or vinculum. Thus, $4b+(b-c)$, or $4b+\overline{b-c}$, indicate that $b-c$ is to be added to $4b$.

Remarks.

32. It will be noticed that the term addition in Algebra, is used in an extended sense, the operation being often arithmetical substraction. The addition of a negative quantity is, in fact, the same as the subtraction of the same quantity regarded as positive. The use of negative quantities in Algebra constitutes one of its marked differences from Arithmetic, in which the numbers employed are always supposed to be positive. By reference to the examples, another remarkable distinction between Algebra and Arithmetic will be observed; each sum indicates what quantities were added together, whilst an arithmetical sum contains no trace of the numbers employed. Thus, the sum of 10 and 5 is 15, but the result does not point out the numbers that were added, for 15 might proceed from the addition of 14 and 1, 12 and 3, 13 and 2, &c.

It will be seen hereafter, that in all algebraical operations, the result contains some, if not all, the quantities employed in the investigations.

SUBTRACTION.

33. Subtraction is taking the difference between two or more quantities, and may be regarded as the undoing of a previous addition.

If we were required to subtract $+ 6 - 4$, or 2 from 12, the result plainly ought to be 10. But if we write $6 - 4$ beneath the 12, and perform the subtraction of each term separately, the subtraction of 6 from 12 will give a remainder 6, which is too small by 4, because we were not required to take 6, but 2 from 12. We can only correct the error by adding $+ 4$, and, hence, we have $12 - (+6 - 4) = 12 - 6 + 4 = 10$, as before. We observe, in the last result, that the signs of 6 and 4 have both been changed.

Again, let it be required to subtract $+ b - c$ from a. The subtraction of b from a will be indicated by writing $a - b$, but this remainder is too small by c, because we were not required to subtract b itself from a, but what remained of b after it was reduced by c. The error can only be corrected by adding plus c to the result. Hence, $a - (+ b - c) = a - b + c$.

In this general example, we see that all the signs of the subtrahend have been changed, and then, it has been added, as in addition. Hence, we have the following

RULE.

Conceive the signs of all the terms of the subtrahend to be changed, and then add up these terms as in addition.

EXAMPLES.

$$\begin{array}{r} -\ 6a + 4b \\ +\ 4a - \ b \\ \hline -10a + 5b \end{array} \qquad \begin{array}{r} 3\sqrt{x} + 5a^2 \\ 6\sqrt{x} - 8a^2 \\ \hline -3\sqrt{x} + 13a^2 \end{array} \qquad \begin{array}{r} 7x^2 + \ m \\ 7x^2 + 2m \\ \hline 0 - \ m \end{array} \qquad \begin{array}{r} 2\sqrt{a} + \ ax \\ 4\sqrt{a} + 6ax \\ \hline -2\sqrt{a} - 5ax \end{array}$$

$$\begin{array}{ll} \text{From} & \sqrt{x} + b - 10 + 8a^2 + 76y + 18a + 19x + 11n \\ \text{Take} & d - 20 + 9x + 5n - 2\sqrt{x} + 3b + 4a^2 + z + 2s \\ \hline & 3\sqrt{x} - 2b + 10x + 4a^2 + 18a + 6n - d - z - 2s + 76y + 10 \end{array}$$

In the third example we have written zero as the difference between $7x^2$ and $7x^2$, but it is more usual to denote zero by a blank.

$$\begin{array}{ll} \text{From} & 3xy + 14\sqrt{y} + z \quad + 2n - w \\ \text{Take} & 3xy + z - 11 + 11\sqrt{y} - 2n + w \\ \hline & \quad 11 \quad + 3\sqrt{y} + 4n - 2w \end{array}$$

$$\begin{array}{ll} \text{From} & 12a^2 + 4bx - cx^2 + m^2 + 4b^2 - c^2x^2 + ns \\ \text{Take} & 4b^2 - c^2x^2 - 6bx + cx^2 - m^2 - 3ns + 10b^2 - wy + 12a^2 \\ \hline & 10bx - 2cx^2 + 2m^2 + 4ns - 10b^2 + wy \end{array}$$

34. The remainder added to the subtrahend ought to be equal to the minuend, and the result, therefore, can be verified, as in Arithmetic.

It will be seen that an Algebraic subtraction does not necessarily imply diminution, and that the remainder may not only exceed the subtrahend and minuend separately, but may be equal to their arithmetical sum. In general, the subtraction of a negative quantity is equivalent to the addition of the same quantity taken positively. Thus, $-b$ taken from a gives $a + b$ for a remainder. This subject may be illustrated by a simple example: Suppose a to represent the value of an estate exclusive of its liabilities, and b a mortgage upon that estate, then $a - b$

will represent the actual value of the estate. Now, suppose some friend of the owner should determine to cancel his debt; he could do this, either by removing the mortgage (in algebraical language, taking away $-b$), or by giving him a sum of money equal to the debt, that is, a sum represented by $+b$. So we see that, taking away $-b$ is equivalent to adding $+b$.

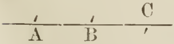
35. Quantities are considered negative, when opposed in character or direction to other quantities of the same kind, that are assumed to be positive.

Thus, if a ship leave port with the intention of sailing due north, but encounters adverse winds, and is driven south part of the time; to get the distance passed over north, we must obviously subtract the distance sailed over south. If then the direction north be considered positive, the direction south must be considered negative. Suppose the vessel sails first day 100 miles north to 10 south, second day 80 miles north and 30 south, the entire distance passed over in a northern direction will be expressed by $100 + 80 - 10 - 30 = 140$.

If a man agree to labor for a dollar per day, and to forfeit half a dollar for every idle day, and he labor 4 days and is idle 2, his wages will be $4 \times 1 - 2 \times \frac{1}{2}$, or $4 \times 1 + 2\,(-\frac{1}{2}) = 3$. We see that we have regarded as negative, either the forfeiture as opposed to gain, or the idle days as opposed to the working days.

If a man's age be now thirty years, his age four years hence will be expressed by $30 + 4$; his age four years ago by $30 - 4$. Here future time being positive, past time is negative. If a steamboat sail with a velocity of 10 miles per hour, and encounter a head wind that would carry it back at the rate of 8 miles per hour, then its rate of advance will be expressed by $10 - 8$, or 2 miles per hour. But if it be carried back at the rate of 12 miles per hour, then its rate of advance will be expressed by $10 - 12$, or -2 miles per hour. It will then plainly be carried back, and the minus indicates a change of direction.

Distance, when estimated as positive in one direction, is considered negative in the opposite direction. Thus, let the distance $AB = n$ and $BC = m$,



, the distance BC being considered positive on the right of B. Then $AC = m + n$. Now suppose the distance BC be estimated on the left of B, the point C falling between B and A, then $AC = m - n$. We see that the distance BC, which was regarded as positive on the right of B, became negative when estimated on the left.

In general, the minus sign may be regarded as always indicating either that a quantity has changed its character or direction, or that it is just the opposite of something else of the same kind that is considered positive.

MULTIPLICATION.

36. Multiplication is a short method of adding the multiplicand to itself as many times as there are units in the multiplier. Thus, to multiply a by b is to add a to itself b times, and since the addition of a to itself twice is $2a$, three times $3a$, &c., the addition of a to itself b times will be ba. Hence the product of a by b will be ba or ab, for it plainly matters not in what order we write the factors.

There are three cases:—

CASE I.

37. When both multiplicand and multiplier are simple quantities.

Before giving a rule for the multiplication, it will be necessary to show that the product of simple quantities having like signs, both plus or both minus, is always positive, and that the product is always negative when they have unlike signs. The product of $+a$ by $+b$ is plainly $+ab$, because plus a added to itself b times must, of course, be positive.

And $-a$ by $+b$ must be negative, because $-a$ added to itself any number of times must, of course, retain its sign. But $-a$ by $-b$ will give $+ab$, for the b having a minus sign before it, indicates the reverse of what it did before, and therefore denotes that $-a$ is to be subtracted from itself b times. But we have seen that the subtraction of $-a$ is the same as the addition of $+a$; in like manner, the subtraction of $-a$ twice is the same as the addition of $+2a$; the subtraction of $-a$ three times the same as the addition of $+3a$; and the subtraction of $-a$, b times, the same as the addition of $+a$, $+b$ times, or $+ab$. Hence, the product $-a$ by $-b$ is $+ab$, and we see that like signs produce plus, and unlike signs, minus.

Let it now be required to multiply a^2 by a^4. The exponents indicate that a enters twice as a factor in a^2, and four times as a factor in a, hence, it will enter 6 times as a factor in the result.

Hence, $a^2 \times a^4 = aaaaaa = a^6$, and we see that the multiplication is effected by adding the exponents of the same letter.

To multiply a^2 by ab is the same as multiplying a^2 by a, and then multiplying the result by b. But to multiply a^2 by a, we have only to add the exponents of a^2 and a, hence the result is a^3. Now multiply by b, and we have a^3b.

The multiplication is effected by adding the exponents of like letters, and writing after the result the letter which is not common to the multiplicand and multiplier, with its primitive exponent. In like manner $a^2 \times ab^3$ will be a^3b^3. To multiply a^2c by ab, is the same as multiplying a^2c by a, and then the result by b. But a^2c by a, as we have seen, gives a^3c, and that product by b, will plainly be a^3cb. And the multiplication is again effected by adding the exponents of like letters, and writing after the result the letters, which are not common, affected with their primitive exponents. The same reasoning can be extended to any number of factors, the multiplication being effected in every instance, by adding the exponents of like letters, and writing after the result, the letters not common, with their primitive exponents. Thus, a^2cd by ab gives a^3cdb, and a^2cd by abm gives a^3cdbm. We have taken monomials whose coefficients were unity. If we were required to multiply a by $2b$, the product of a by b has to be multiplied by 2, but the product a by b is ab, hence the result is $2ab$. If we were required to multiply $2a$ by b, then $2a$ has to be added to itself as many times as there are units in b, but to add $2a$ to itself twice gives $2a \times 2$ or $4a$, to add it three times gives $2a \times 3$, or $6a$, and to add it b times gives $2a \times b$, or $2ab$. In like manner, if we were required to multiply $2a$ by $4b$, then $2a$ has to be added to itself $4b$ times, and the result will plainly be $2a \times 4b$, or $8ab$: the same result that we would get by multiplying the literal factors together, and prefixing the product of the coefficient for a new coefficient. Hence we have for the multiplication of monomials, the following

RULE.

38. *Multiply the coefficients together for a new coefficient, and write after it all the literal factors, common to the two monomials, affected with exponents equal to the sum of the exponents in the multiplicand and multiplier, and those, which are not common, with their primitive exponents. If the monomials have like signs, give the plus sign to the product; if they have unlike signs, give the minus sign to the product.*

3

EXAMPLES.

1. a^2bc	by abc.	*Ans.* $a^3b^2c^2$.
2. $-a^2bc$	by abc.	*Ans.* $-a^3b^2c^2$
3. a^2bc	by $-abc$.	*Ans.* $-a^3b^2c^2$.
4. $2a^2bc$	by abc.	*Ans.* $2a^3b^2c^2$.
5. $2a^2bc$	by abc.	*Ans.* $-2a^3b^2c^2$.
6. $2a^2bc$	by $-abc$.	*Ans.* $-2a^3b^2c^2$.
7. a^2bc	by $2abc$.	*Ans.* $2a^3b^2c^2$.
8. a^2bc	by $-2abc$.	*Ans.* $-2a^3b^2c^2$.
9. a^mb^n	by $a^{m+1}b^{n+1}$.	*Ans.* $a^{2m+1}b^{2n+1}$.
10. $a^{+m}b^n$	by $a^{m-1}b^{n-1}$	*Ans.* $a^{2m-1}b^{2n-1}$.
11. $a^{-m}b^{-n}c^{-n}$	by $a^mb^{-2}c^2$.	*Ans.* $a^0b^{-n-2}c^{-n\ \ 2}$.
12. $4a^{-m}b^nc$	by $-8a^mb^{-3}c^{-4}$.	*Ans.* $-32a^0b^{n-3}c^{-3}$.
13. $4a^3b^3c^3$	by $-8a^4b^4c^4$	*Ans.* $-32a^7b^7c^7$.
14. $6a^{-3}b^{-3}c^{-3}$	by $7a^{-4}b^{-4}c^{-4}$.	*Ans.* $42a^{-7}b^{-7}c^{-7}$.
15. $3a^6b^6c^6$	by $9a^{-2}b^{-2}c^{-2}$.	*Ans.* $27a^4b^4c^4$.
16. $x^my^nz^p$	by $4xyzb$.	*Ans.* $4x^{m\ \ 1}y^{n+1}z^{p+1}b$.
17. x^myzc	by ac.	*Ans.* $+x^myzac^2$.
18. x^py^2c	by xy.	*Ans.* $x^{p\ \ 1}y^3c$.
19. x^pyc	by xy^2.	*Ans.* $x^{p+1}y^3c$.
20. $x^{p-1}yc$	by x^2y^2.	*Ans.* $x^{p+1}y^3c$.
21. $x^{p-2}y^3c^{-2}$	by c^3x^3.	*Ans.* $x^{p+1}y^3c$.

Remarks.

39. We see, from the last four examples, that we can make any change in the position of the exponents, provided we retain the factors, and get the same result.

We see, from examples 13, 14, and 15, that when the factors of the multiplicand and multiplier are homogeneous, the factors of the result will also be homogeneous.

Example 11 shows that this will not be true when the multiplicand is alone made up of homogeneous factors; and Example 1 shows that it will not be true when the multiplier alone is homogeneous.

40. Any number of monomials may be multiplied together, in accordance with the preceding principles.

EXAMPLES.

1.	$a^2c \times$	$ab \times$	$bd \times mn.$	*Ans.*	$a^3b^2dcmn.$
2.	$-a^2c \times$	$-ab \times$	$bd \times mn.$	*Ans.*	$a^3b^2dcmn.$
3.	$-a^2c \times$	$-ab \times$	$-bd \times mn.$	*Ans.*	$-a^3b^2dcmn.$
4.	$-a^2c \times$	$-ab \times$	$bd \times -mn.$	*Ans.*	$-a^3b^2dcmn.$

We see, that when the monomials are all positive, the result will be positive; and when the negative monomials are even in number, the result will also be positive. But when an odd number of negative monomials are multiplied together, as in the fourth Example, or when an odd number of negative monomials are multiplied by a positive monomial, as in Example 3, the result will be negative. The result would also be negative, if an odd number of negative monomial factors was multiplied into any number of positive monomial factors.

CASE II.

41. When the multiplicand is a compound quantity and the multiplier a monomial.

The multiplicand is to be repeated as many times as there are units in the multiplier; each term of the multiplicand is then to be multiplied by the multiplier, and the partial products to be connected with their appropriate signs. We might assume, what has already been proved for monomials, that the product of a positive quantity by a positive, or of a negative quantity by a negative, gives a positive result; and that the product of a positive quantity by a negative gives a negative result. But we can demonstrate this rule more rigorously: Let it be required to multiply $a - a$ by $+ c$. We know that $a - a$ is zero, and that zero repeated c times must still be zero. Hence, the product of $a - a$ by $+ c$ must be zero. But $+ a$ by $+ c$ gives $+ ac$ for a product, and, therefore, the product of $- a$ by $+ c$ must be $- ac$, in order to cancel the first product. Hence, the product of a negative quantity by a positive gives a negative result. Or, to express the whole algebraically,

$$\begin{array}{r} a - a = o \\ \underline{+ c \quad} \\ +ac - ac = o \end{array}$$

Let it now be required to multiply $a - a$ by $-c$. The multiplicand being zero, the product must be zero. But, from what has just been shown, the product of $+a$ and $-c$ is $-ac$, hence, the product of $-a$ by $-c$ must be $+ac$, to destroy the first product. Or, in other words, the product of a negative quantity by a negative must be effected with the positive sign.

$$\text{Algebraically,} \quad \begin{array}{r} a - a = o \\ -c \\ \hline -ac + ac \end{array}$$

We conclude that the product of a positive quantity by a positive, and of a negative quantity by a negative, is positive; and that the product of a positive quantity by a negative is negative. Briefly, we say like signs give a positive result, and unlike, a negative result.

RULE.

Multiply each term of the multiplicand by the multiplier, and connect the results with their appropriate signs.

EXAMPLES.

$a^2 + b$	by a.	*Ans.* $a^3 + ab$.
$a^2 - b$	by a.	*Ans.* $a^3 - ab$.
$-a^2 + b$	by a.	*Ans.* $-a^3 + ab$.
$-a^2 - b$	by $-a$.	*Ans.* $a^3 + ab$.
$-a^2 + b$	by $-a$.	*Ans.* $a^3 - ab$.
$a^2b + c + d$	by a.	*Ans.* $a^3b + ac + ad$.
$a^2b - c + d$	by a.	*Ans.* $a^3b - ac + ad$.
$a^2b + c - d$	by a.	*Ans.* $a^3b + ac - ad$.
$-a^2b + c + d$	by a.	*Ans.* $-a^3b + ac + ad$.
$-a^2b - c + d$	by a.	*Ans.* $-a^3b - ac + ad$.
$-a^2b - c - d$	by a.	*Ans.* $-a^3b - ac - ad$.
$-a^2b - c - d$	by $-a$.	*Ans.* $a^3b + ac + ad$.
$-a^2b - c + d$	by $-a$.	*Ans.* $a^3b + ac - ad$.
$-a^2b + c + d$	by $-a$.	*Ans.* $a^3b - ac - ad$.
$a^2b + c + d$	by $-a$.	*Ans.* $-a^3b - ac - ad$.
$m^2n + a + 2c$	by s.	*Ans.* $m^2ns + as + 2cs$.
$m^2n + a + 2c$	by $3c$	*Ans.* $3cm^2n + 3ac + 6c^2$
$m^2n + a + 2c$	by $4mn$.	*Ans.* $4m^3n^2 + 4amn + 8mnc$.
$8m - 6a + 11c$	by $2c^3$.	*Ans.* $16mc^3 - 12ac^3 + 22c^4$.

CASE III.

42. When the multiplicand and multiplier are both compound quantities.

The multiplicand, as in the two preceding cases, is to be repeated as many times as there are units in the multiplier. The multiplicand and multiplier will, in general, be made up of some positive and some negative terms. Let a denote the sum of all the positive terms in the multiplicand, and b the sum of all the negative terms. Let c denote the sum of the positive terms in the multiplier, and d the sum of all the negative terms. We write the multiplier beneath the multiplicand and multiply all the terms of the one by all the terms of the other.

$$
\text{Thus,}\quad \begin{array}{llll}
a & -\,b & & \\
c & -\,d & & \\
\hline
ac & -\,bc & & \\
 & -\,ad & +\,bd & \\
\hline
ac & -\,bc & -\,ad & +\,bd.
\end{array}
$$

To explain this result, let us drop for a moment the consideration of $-d$, then $a - b$ must be repeated c times. From what has been shown, the result of this multiplication will be $ac - bc$. But we were not required to multiply $a - b$ by c alone, but by c after it had been diminished by d; hence, the result, $ac - bc$, is too great by $a - b$ taken d times. To correct the error, then, we must subtract the product of $a - b$ by d from the first product $ac - bc$. The product of $a - b$ by $+ d$, from what has been shown, will be $+ ad - bd$, and to indicate that this must be subtracted, we write it in parenthesis with the minus sign before it. The whole result will be $ac - bc - (+ ad - bd) = ac - bc - ad + bd$, since the signs of the quantities subtracted must be changed. By examining the result, we will observe, as before, that the product of quantities affected with like signs is positive, and that of quantities affected with unlike signs is minus.

43. It is found most convenient to arrange both polynomials with reference to the highest or lowest exponent of the same letter. Thus, $x^3 + x^2 + x + a$ is arranged with reference to the highest exponent of x; and $a + x + x^2 + x^3$ is arranged with reference to the lowest exponent of the same letter. In these expressions, x is supposed to enter to the zero power in the term a, as will be explained under the head of Division.

RULE.

44. *Arrange the multiplicand and multiplier with reference to the highest or lowest exponent of the same letter (if they have a common letter), and then multiply each term of the one polynomial by each term of the other polynomial, beginning on the left. Set down the result of the multiplication of the second term of the multiplier under that of the first term, only removed one place further to the right, and the result of the multiplication of the third term of the multiplier under that of the second, and continue the operation until the multiplication is complete. Then reduce the whole result to the simplest form.*

EXAMPLES.

1.
$$\begin{array}{l} x+a \\ x-a \\ \hline x^2+ax \\ \quad\; -ax-a^2 \\ \hline x^2+o-a^2=x^2-a^2. \end{array}$$

2.
$$\begin{array}{l} x^2+xy+a \\ x\;+y \\ \hline x^3+x^2y+ax \\ \quad\;+x^2y+xy^2+ay \\ \hline x^3+2x^2y+ax+xy^2+ay. \end{array}$$

3.
$$\begin{array}{l} x^3+ax^2-x \\ x^2+ax+1 \\ \hline x5+ax^4-x^3 \\ \quad\;+ax^4+ax^3-ax^2 \\ \quad\quad\quad+x^3+ax^2-x \\ \hline x5+2ax^4+ax^3+o-x \end{array}$$

4.
$$\begin{array}{l} x^2+2xy+y^2 \\ \quad\quad x-y \\ \hline x^3+2x^2y+y^2x \\ \quad\;-x^2y-2y^2x-y^3 \\ \hline x^3+x^2y-y^2x\;\;-y^3 \end{array}$$

45. It will be observed in the above examples, that by arranging with reference to a certain letter, and by removing each result one place further to the right than the preceding, the terms which reduce fall immediately the one below the other. And it is to save the trouble of looking for the terms which reduce, that this arrangement and system are made. It will also be noticed that in every product there are two terms which do not reduce with other terms, viz., those which result from the multiplication of the quantities affected with the highest and lowest exponents of the arranged letter.

In the first example, x^2 and $-a^2$ are the irreducible terms, in the second, x^3 and ay.

EXAMPLES.

6. Multiply $x + y$ by $x + y$.

Ans. $x^2 + 2xy + y^2$.

7. Multiply $x + y$ by $x - y$.

Ans. $x^2 - y^2$.

8. Multiply $x^2 + 2xy + y^2$ by $+ x + y$.

Ans. $x^3 + 3x^2y + 3y^2x + y^3$.

9. Multiply $x^2 - 2xy + y^2$ by $x + y$.

Ans. $x^3 - x^2y - xy^2 + y^3$.

10. Multiply $x^4 - y^4$ by $x^2 + y^2$.

Ans. $x^6 + y^2x^4 - x^2y^4 - y^6$.

11. Multiply $a^2 + 2ax + x^2$ by $x + a$.

Ans. $x^3 + 3ax^2 + 3a^2x + a^3$.

12. Multiply $x^3 + a^3 + 3ax + 3ax^2$ by $x + a$.

Ans. $x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$.

13. Multiply $7x^{\frac{1}{2}} + 2a$ by $3y^{\frac{1}{2}}$.

Ans. $21x^{\frac{1}{2}}y^{\frac{1}{2}} + 6ay^{\frac{1}{2}}$.

14. Multiply $4x^{\frac{1}{3}} + y^{\frac{1}{3}}$ by $4x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

Ans. $16x^{\frac{2}{3}} - y^{\frac{2}{3}}$.

15. Multiply $x^3 + ax^2 + a^2x + a^3$ by $2a + 2x$.

Ans. $2x^4 + 4ax^3 + 4a^2x^2 + 4a^3x + 2a^4$.

16. Multiply $x^2 + xy + y^2$ by $7x + 7y$.

Ans. $7x^3 + 14x^2y + 14xy^2 + 7y^3$.

17. Multiply $x^4 + ax^3 + a^2x^2 + a^3x + a^4$ by $4x + 4a$.

Ans. $4x^5 + 8ax^4 + 8a^2x^3 + 8a^3x^2 + 8a^4x + 4a^5$.

18. Multiply $2x^2 + 2ax + 2a^2$ by $3x + 3y$.

Ans. $6x^3 + 6ax^2 + 6x^2y + 6a^2x + 6axy + 6a^2y$.

19. Multiply $2a^2 - 2b^2$ by $2a + 2b$.

Ans. $4a^3 + 4a^2b - 4ab^2 - 4b^3$.

20. Multiply $2x + 3a + 4b$ by $y + z$.

Ans. $2xy + 3ay + 4by + 2xz + 3az + 4bz$.

21. Multiply $2x + 3a + 4b$ by $2x - 2y$.

Ans. $4x^2 - 4xy + 6ax + 8bx - 6ay - 8by$.

22. Multiply $4y^2 + 4x^2 + 4mn$ by $2x + 2y + 2z$.

Ans. $8y^3 + 8yx^2 + 8ymn + 8x^3 + 8xy^2 + 8xmn + 8y^2z + 8x^2z + 8mnz$.

Remarks.

46. It will be seen, by inspecting the above results, that when the two polynomials are both homogeneous, the product is also homogeneous, and of a degree equal to the sum of the dimensions of the multiplicand and multiplier. Thus, in Example 7th, the first polynomial is homogeneous, and of the first degree; and the second is also homogeneous of the first degree, the product is homogeneous of the second degree. In Example 8th, the multiplier is homogeneous of the second degree, and the multiplier is homogeneous of the first degree; the product is homogeneous of the third degree. The same may be noticed in several other examples.

2d. We notice also that when the coefficients are the same in each term of the multiplicand, and the same in each term of the multiplier, and all the terms of both polynomials are positive, that the sum of the coefficients in the product will be equal to the product arising from multiplying the sum of the coefficients in one polynomial by the sum of the coefficients in the other. Thus, in Example 16, each coefficient of the multiplicand is 1, and each coefficient of the multiplier 7. The sum of the coefficients of the multiplicand is 3, and that of the multiplier 14: the sum of the coefficients in the product is $3 \times 14 = 42$. The same law may be noticed in Examples 15, 18, and 22.

47. Examples 10 and 19 show that when the multiplicand is composed of the difference of two terms, whose coefficients are equal, the algebraic sum of the coefficients in the product is zero. Examples 7 and 21 show that this sum is also zero when the multiplier is composed of two terms with contrary signs and equal coefficients.

3d. It has already been remarked that there are always two terms, which do not reduce with any other terms. We can only reduce similar terms, and when the two polynomials have been arranged with respect to a certain letter, the products of the extreme terms are dissimilar to the other partial products. The whole process of division of polynomials is based upon this fact, and it ought to be remembered. By attending to the above laws in regard to the product, we can often by a simple inspection detect errors in the multiplication.

There are three theorems of great importance, which must be committed to memory.

Theorem I.

48. The square of the sum of two quantities is equal to the square of the first, plus the double product of the first by the second, plus the square of the second.

Let x denote the first quantity, and a the second, then $x + a =$ sum and $(x + a)^2 = (x + a)\ (x + a)$, which, by performing the multiplication, will be found equal to $x^2 + 2ax + a^2$.

Hence $(x + a)^2 = x^2 + 2ax + a^2$, as enunciated.

So $(10 + 5)^2 = 10^2 + 20 \times 5 + 5^2 = 100 + 100 + 25 = 225$.

And $(40 + 6)^2 = 40^2 + 80 \times 6 + 6^2 = 1600 + 480 + 36 = 2116$.

In like manner $(x^m + a^m)^2 = x^{2m} + 2x^m a^m + a^{2m}$.

The rule may be extended to binomials of any form.

Thus, $(3a + b)^2 = (3a)^2 + 6ab + b^2 = 9a^2 + 6ab + b^2$.

$(7x + 5y)^2 = (7x)^2 + 14x\ (5y) + (5y)^2 = 49x^2 + 70xy + 25y^2$.

A polynomial may be squared by the same formula.

Let it be required to square $x + a + y$.

Make $x + a = z$.

Then $x + a + y = z + y$ and $(x + a + y)^2 = (z + y)^2 = z^2 + 2zy + y^2$.

Now substitute for z its value $x + a$, and we have $(x + a + y)^2 = (x + a)^2 + 2\ (x + a)\ y + y^2 = x^2 + 2ax + x^2 + 2xy + 2ay + y^2$.

Required the square of $2y + x + m + n$.

Let $2y + x = z$, and $m + n = s$.

Then $(2y + x + m + n)^2 = (z + s)^2 = z^2 + 2zs + s^2 = (2y + x)^2 + 2\ (2y + x)\ (m + n) + (m + n)^2 = 4y^2 + 4yx + x^2 + 4ym + 4yn + 2xm + 2xn + m^2 + 2mn + n^2$.

Required the square of $3y + 4x + m + 2n + 5$.

Represent the first three terms by a single letter, and the last two also by a single letter, and proceed as before.

Theorem II.

49. The square of the difference of two quantities is equal to the square of the first, minus the double product of the first by the second, plus the square of the second.

Let x and a denote the quantities, then $x - a =$ difference.

And $(x - a)^2 = (x - a)\ (x - a) = x^2 - 2ax + a^2$, as enunciated.

So $(10 - 5)^2 = 10^2 - 20 \times 5 + (5)^2 = 100 - 100 + 25 = 25$.

And $(40 - 6)^2 = (40)^2 - 80 \times 6 + 6^2 = 1600 - 480 + 36 = 1156$.

In like manner $(x^m - a^m)^2 = x^{2m} - 2x^m a^m + a^{2m}$.

Required the square of $2x - a$. *Ans.* $4x^2 - 4ax + a^2$.

Required the square of $3x + b - a$.

Ans. $(3x + b)^2 - 2(3x + b)a + a^2 = 9x^2 + 6bx + b^2 - 6ax - 2ab + a^2$.

Required square of $x^3 - a^3$. *Ans.* $x^6 - 2x^3a^3 + a^6$.

Theorem III.

50. The sum of two quantities multiplied by the difference of the same quantities, is equal to the difference of their squares.

Let x and a denote their quantities.

Then $x + a =$ sum, and $x - a =$ difference, and $(x + a)(x - a) = x^2 - a^2$, by performing the multiplication indicated.

So $(10 + 5)(10 - 5) = (10)^2 - (5)^2 = 100 - 25 = 75$.

And $(40 + 6)(40 - 6) = (40)^2 - (6)^2 = 1600 - 36 = 1564$.

Multiply $(4a + 6)$ by $(4a - 6)$. *Ans.* $16a^2 - 6^2$.

Multiply $7a + b + c$ by $7a + b - c$.

Ans. $(7a + b)^2 - c^2 = 49a^2 + 14ab + b^2 - c^2$.

Multiply $x + y + z + m$ by $x + y + z - m$.

Ans. $(x + y + z)^2 - m^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz - m^2$.

Multiply $x + y + z + m$ by $x + y - z - m$.

Ans. $(x + y)^2 - (z + m)^2 = x^2 + 2xy + y^2 - z^2 - 2mz - m^2$

Remarks.

51. If we omit the exponents of the extreme terms in the expression, $x^2 + 2ax + a^2$, and connect these terms with the exponents so omitted, by the sign of the middle term $2ax$, we have $x + a$, the binomial, which squared gave $x^2 + 2ax + a^2$. In like manner, if we omit the exponents of x^2 and a^2 in the expression, $x^2 - 2ax + a^2$, and connect them with their exponents omitted by the sign of the middle term, we have $x - a$, the binomial, which squared gave $x^2 - 2ax + x^2$.

In like manner, if we have given the difference of two squares, we can readily determine the quantities which, by being multiplied together, gave this difference.

Thus $m^2 - n^2 = (m + n)(m - n)$, and $r^2 - s^2 = (r + s)(r - s)$.

Omit the exponents, connect the terms by the sign plus, and we have the sum of the two quantities; omit and connect by the sign minus, and we have the difference. And the product of the sum by the difference, is the difference of the squares. These remarks are preliminary to an important subject.

FACTORING POLYNOMIALS.

52. It is often a matter of great importance to resolve a polynomial into its factors. The reduction of expressions can often be effected in this way, and in no other. Practice alone can make the student expert in the resolution of an algebraic expression into its factors. A few rules may, however, assist the beginner.

53. 1st. Look out for the terms which have a common factor, and write them in parenthesis, as a multiplier of that factor; next look for another set of terms also having a common factor, and write them in like manner; proceed thus until all the terms are taken up. Observe whether the parenthetical expressions are the same, if so, multiply the common parenthesis into the *algebraic* sum of the terms to which it serves as a coefficient.

EXAMPLES.

1. $bx + ba + cx + ca = b(x + a) + c(x + a) = (x + a)(b + c)$.
2. $bx + ba - cx - ca = b(x + a) + (x + a)(-c) = (x + a)(b - c)$.
3. $-bx - ba + cx + ca = (-b)(x + a) + (x + a)c = (x + a)(c - b)$.
4. $bx + cx + ba + dx + da + ca = (x + a)b + (x + a)c + (x + a)d = (x + a)(b + c + d)$.
5. $bx + cx + ba - dx - da + ca = (x + a)(b + c - d)$.
6. $x^2 + ax + bx + ab = (x + a)x + (x + a)b = (x + a)(x + b)$.
7. $x^2 - ax + bx - ab = (x - a)(x + b)$.
8. $x^2 - ax + bx - ab + x^3 - ax^2 = (x - a)(x + b + x^2)$.
9. $x^2 - ax + bx - ab + mx^3 - amx^2 = (x - a)(x + b + mx^2)$.
10. $x^2 - ax + bx - ab + amx^2 - mx^3 = (x - a)(x + b - mx^2)$.

54. 2d. After having found a common factor or parenthesis, see whether that factor may not admit of farther reduction. (Articles 48, 49, and 50.)

EXAMPLES.

1. $bn^2 + 2bn + b + cn^2 + 2cn + c = b\ (n^2 + 2n + 1) + c\ (n^2 + 2n + 1) = (b + c)\ (n^2 + 2n + 1) = (b + c)\ (n + 1)^2$. (Article 48.)

2. $bn^2 - 2bn + b + cn^2 - 2cn + c = (b + c)\ (n - 1)^2$. (Art. 49).

3. $am^2 + bm^2 - an^2 - bn^2 = (a + b)\ (m^2 - n^2) = (a + b)\ (m + n)\ (m - n)$. (Article 50).

4. $am^2 - bm^2 - an^2 + bn^2 = (a - b)\ (m + n)\ (m - n)$.

5. $n^3 + 2n^2 + n = n\ (n + 1)^2$.

6. $n^3 + 2n^2 = n^2\ (n + 2)$. Admits of no lower reduction.

7. $n^3 - 2n^2 + n = n\ (n - 1)^2$.

8. $n^3 - 2n^2 + n + mn^2 - 2mn + m = (n + m)\ (n - 1)^2$.

9. $n^3 - 2n^2 + n - mn^2 + 2mn - m = (n - m)\ (n - 1)^2$.

10. $-n^3 + 2n^2 - n + mn^2 - 2mn + m = (m - n)\ (n - 1)^2$.

11. $-n^3 + 2n^2 - n - mn^2 + 2mn - m = -(n + m)\ (n - 1)^2$.

12. $m^2 - n^2 - 2cn - c^2 = m^2 - (n + c)^2$.

We have now the differences of two squares, and to apply the formula

$$x^2 - a^2 = (x + a)\ (x - a). \quad \text{(Art. 50).}$$

We see that $m^2 = x^2$, or $m = x$, and $(n + c)^2 = a^2$, or $n + c = a$. Hence $(x + a) = (m + n + c)$, and $(x - a) = (m - n - c)$. Therefore $m^2 - (n + c)^2 = (m + n + c)\ (m - n - c)$.

13. $m^2 - n^2 + 2cn - c^2 = m^2 - (n - c)^2 = (m + n - c)\ (m + c - n)$. In this Example $m = x$, and $n - c = a$.

14. $m^2 + 2bm + b^2 - n^2 - 2cn - n^2 = (m + b)^2 - (n + c)^2 = (m + b + n + c) \times (m + b - n - c)$.

55. 3d. Expressions may sometimes be thrown into factors by an artifice, when they do not at first glance appear to admit of resolution into factors. One of the simplest contrivances to effect a decomposition into factors is the addition to, and subtraction of the same quantity from the given expression, which operation does not, of course, alter its value.

EXAMPLES.

1. $x^2 - 2x - 15 = x^2 - 2x + 1 - 15 - 1 = x^2 - 2x + 1 - 16 = (x - 1)^2 - 16 = (x - 1)^2 - (4)^2 = (x - 1 + 4)\ (x - 1 - 4)$. Art. 50. $= (x + 3)\ (x - 5)$.

In this example, the decomposition was effected by adding + 1, and subtracting the same from the given expression. The first three terms of the equivalent expression were thus made a perfect square $= (x - 1^2)$, and the whole was made the difference between two squares.

2. $x^2 + 4x - 12 = x^2 + 4x + 4 - 12 - 4 = (x + 2)^2 - (4)^2 = (x + 6)(x - 2)$.
3. $x^2 + 2x - 8 = x^2 + 2x + 1 - 8 - 1 = (x + 1)^2 - (3)^2 = (x + 4)(x - 2)$.
4. $x^2 + 4x - 21 = x^2 + 4x + 4 - 21 - 4 = (x + 2)^2 - (5)^2 = (x + 7)(x - 3)$.
5. $x^2 + 8x - 48 = x^2 + 8x + 16 - 48 - 16 = (x + 4)^2 - (8)^2 = (x + 12)(x - 4)$.
6. $x^2 + 10x + 24 = x^2 + 10x + 25 + 24 - 25 = (x + 5)^2 - (1)^2 = (x + 6)(x + 4)$.
7. $x^2 + 8x + 12 = x^2 + 8x + 16 + 12 - 16 = (x + 4)^2 - (2)^2 = (x + 6)(x + 2)$.
8. $x^2 + 20x + 84 = x^2 + 20x + 100 + 84 - 100 = (x + 10)^2 - (4)^2 = (x + 14)(x + 6)$.
9. $x^2 + 12x + 2 = x^2 + 12x + 36 + 2 - 36 = (x + 6)^2 - \sqrt{(34)}^2 = (x + 6 + \sqrt{34})(x + 6 - \sqrt{34})$.
10. $x^2 + 3x + 5 = x^2 + 3x + \frac{9}{4} + 5 - \frac{9}{4} = (x + \frac{3}{2})^2 - \sqrt{(-\frac{11}{4})}^2 = (x + \frac{3}{2} + \sqrt{-\frac{11}{4}})(x + \frac{3}{2} - \sqrt{-\frac{11}{4}})$.
11. $x^2 - 10x + 24 = x^2 - 10x + 25 + 24 - 25 = (x - 5)^2 - (1)^2 = (x - 4)(x - 6)$.
12. $x^2 - 8x + 12 = x^2 - 8x + 16 + 12 - 16 = (x - 4)^2 - (2)^2 = (x - 2)(x - 6)$.
13. $x^2 - 10x - 24 = x^2 - 10x + 25 - 24 - 25 = (x - 5)^2 - (7)^2 = (x + 2)(x - 12)$.
14. $x^4 - 10x^2 - 24 = x^4 - 10x^2 + 25 - 24 - 25 = (x^2 - 5)^2 - (7)^2 = (x^2 + 2)(x^2 - 12)$.
15. $x^6 + 4x^3 - 12 = x^6 + 4x^3 + 4 - 12 - 4 = (x^3 + 2)^2 - (4)^2 = (x^3 + 6)(x^3 - 2)$.
16. $a^2 + 2ab - 3b^2 = a^2 + 2ab + b^2 - 4b^2 = (a + b)^2 - (2b)^2 = (a + 3b)(a - b)$.
17. $a^2 + 2ab - m^2 - 2bm = a^2 + 2ab + b^2 - m^2 - 2bm - b^2 = (a + b)^2 - (m + b)^2 = (a + m + 2b)(a - m)$.
18. $a^2 + 2ab - m^2 + 2bm = a^2 + 2ab + b^2 - m^2 + 2bm - b^2 = (a + b)^2 - (m - b)^2 = (a + m)(a - m + 2b)$.

DIVISION.

56. Division consists in finding how many times one quantity is contained in another.

The quantity divided is called the *dividend,* that by which it is divided, the *divisor,* and the result obtained the *quotient.*

57. It follows from this that the quotient multiplied by the divisor must give the dividend.

The quotient is said to be *exact* when the dividend contains the divisor an exact number of times. When this is not so, the quotient is called imperfect.

It will be seen that the object of division is to find a quantity called the quotient, which, when multiplied by the divisor, will give the dividend.

The result of the division of $4ax$ by $2a$ is plainly $2x$, because $2a \times 2x = 4ax$, the dividend.

$$\text{So,} \quad \frac{7a^2c}{ac} = 7a, \text{ because } 7a \times ac = 7a^2c.$$

But $\frac{-4ax}{2a}$ gives $-2x$ for a quotient, because $2a \times (-2x) = -4ax$.

$$\text{And } \frac{-7a^2c}{ac} = -7a, \text{ because } -7a \times ac = -7a^2c.$$

58. And, in general, since the quotient into the divisor must give the dividend; when the sign of the quantity to be divided is unlike that by which it is divided, the result will be negative, and when the sign of the quantity to be divided is like that by which it is divided, the result will have the positive sign.

$$\text{Thus,} \quad \frac{-ab}{-b} = +a, \text{ because } +a \times -b = -ab.$$

59. In Division, then, as in multiplication, like signs produce $+$, and unlike signs $-$.

There are three cases in Division.

CASE I.

When the dividend and divisor are both monomials.

60. Divide a^3 by a; we are to find a quantity, which, multiplied by a, will give a^3. This quantity is plainly a^2, because $a^2 \times a = a^3$.

And, since, in multiplication, we add the exponents of like letters, we must, in division, subtract the exponents of like letters.

Divide $6a^3$ by $2a$, the result is obviously $3a^2$; because $2a \times 3a^2 = 6a^3$. And since, in multiplication, we multiply the coefficients together for a new coefficient, we must, in division, divide the coefficients for a new coefficient.

Divide a^2b by a, the result is plainly ab, because $ab \times a = a^2b$. Divide $a^2b^3c^4$ by a, the result is ab^3c^4. And, in general, if there are letters in the dividend not common to the divisor, they will enter into the quotient with their primitive exponents.

RULE.

61. *Divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient. Write after this new coefficient all the letters common to dividend and divisor affected with exponents equal to the excess of the exponents of the dividend over those of the same letters in divisor, and write the letters common to the dividend only with their primitive exponents. Give the quotient the sign* $+$, *where the monomials have like signs, and the sign* $-$, *where they have unlike signs.*

EXAMPLES.

1. Divide x^3y by x. *Ans.* x^2y.
2. Divide $-x^3y$ by x. *Ans.* $-x^2y$.
3. Divide $-x^3y$ by $-x$. *Ans.* x^2y.
4. Divide x^3y by $-x$. *Ans.* $-x^2y$.
5. Divide $4a^mb^pc$ by $2c^2$. *Ans.* $2a^mb^pc^{-1}$.
6. Divide $-40x^my^nz^p$ by $-20x^ny^mz^{2p}$. *Ans.* $2x^{m-n}y^{n-m}z^{-p}$.
7. Divide xy^2 by $x^{\frac{1}{2}}y$. *Ans.* $x^{\frac{1}{2}}y$.
8. Divide $x^{\frac{1}{2}}y$ by xy^2. *Ans.* $x^{-\frac{1}{2}}y^{-1}$.
9. Divide x^3y^5 by x^4y^6. *Ans.* $x^{-1}y^{-1}$.
10. Divide x^4y^6 by x^3y^5. *Ans.* xy.
11. Divide z^2s^3 by z^3s^2. *Ans.* $z^{-1}s$.
12. Divide z^3s^2 by z^2s^3. *Ans.* zs^{-1}.
13. Divide $1000a^mb^n$ by $500a^{-m}b^{-n}$. *Ans.* $2a^{2m}b^{2n}$.
14. Divide $1000a^{-m}b^{-n}$ by $-500a^mb^n$. *Ans.* $-2a^{-2m}b^{-2n}$.
15. Divide $x^{\frac{1}{n}}y^m$ by x^2y^m. *Ans.* $x^{\frac{1}{n}-2}y^{\cdot}$.

62. It will be seen that the division of monomials is impossible when the coefficient of the dividend is not divisible by that of the divisor, and when the divisor contains one or more letters than the dividend.

63. In such cases, the quotient appears in the form of a fraction, which may admit of farther reduction by striking out the common factors. Thus, the quotient arising from the division of $7a^2$ by $2a^2b$, is $\frac{7a^2}{2a^2b}$, because $\frac{7a^2}{2a^2b} \times 2a^2b$ is plainly equal to the dividend $7a^2$. But $\frac{7a^2}{2a^2b}$ may be reduced to $\frac{7}{2b}$ by striking out the common factor, a^2.

In like manner, $7a^2b$, divided by $2ab^2$, is $\frac{7a^2b}{2ab^2} = \frac{7a}{2b}$.

64. We will now demonstrate two principles which will enable us to reduce such expressions still lower.

1st. Any quantity raised to the zero power is equal to unity, that is, $a^0 = 1$. For $\frac{a^m}{a^m}$ by the rule for the exponents in division is equal to $a^{m-m} = a^0$, but any quantity divided by itself is also equal to one.

Hence, $\frac{a^m}{a^m} = a^0$ is also equal to 1. Therefore, $a^0 = 1$.

In like manner, $2^0 = 1$, and $.\ (1000)^0 = 1$, and $(a + b)^0 = 1$, &c.

65. 2d. A quantity may be transferred from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent. For a^{-m} may be multiplied by $\frac{a^m}{a^m} = 1$, without altering its value.

Hence, $$a^{-m} = a^{-m} \times \frac{a^m}{a^m} = \frac{a^0}{a^m} = \frac{1}{a^m}$$

But we know that $a^{-m} = \frac{a^{-m}}{1}$, which we have seen, is also equal to $\frac{1}{a^m}$. The quantity a^{-m} has then gone from the numerator to the denominator, by changing the sign of its exponent.

In like manner, a^{+m} may be multiplied by $\frac{a^{-m}}{a^{-m}} = 1$, without altering its value.

Hence, $$a^m = \frac{a^m}{1} = a^m \times \frac{a^{-m}}{a^{-m}} = \frac{a^0}{a^{-m}} = \frac{1}{a^{-m}}.$$

So, $$\frac{1}{z^p} = \frac{1}{z^p} \times \frac{z^{-p}}{z^{-p}} = \frac{z^{-p}}{z^0} = \frac{z^{-p}}{1} = z^{-p}.$$

The quantity z^p has passed from the denominator to the numerator by changing the sign of its exponent.

So, $$\frac{1}{z^{-p}} = \frac{1}{z^{-p}} \times \frac{z^p}{z^p} = \frac{z^p}{z^0} = \frac{z^p}{1} = z^p.$$

66. By the first principle, the quotient in Example 15 may be changed into $x^{\frac{1}{n}-2} . 1 = x^{\frac{1}{n}-2}$.

By the second principle, the quotient of $7a^2b$ by $2a^2b$ may be changed into $\frac{7}{2}b^{-1}$, and the quotient of $7a^2b$ by $2ab^2$ may be changed into $\frac{7}{2}ab^{-1}$.

Divide x^2y by x^3y^2c. *Ans.* $x^{-1}y^{-1}c^{-1}$, or $\frac{1}{xyc}$.

67. In this example, and in all similar examples, when the divisor contains a letter or letters not contained in the dividend, we may conceive this letter, or these letters, also to enter into dividend raised to the zero power, and to execute the division, we have only to subtract the exponents of like letters, as before. Thus, x^2y may be written x^2yc^0, and we have only to subtract 1, the exponent of c in the divisor, from o, the exponent of c in the dividend. The result will be $o - 1$, or -1.

Divide x^3y by $a^2b^3c^4$; then $x^3y = x^3ya^0b^0c^0$, and the result will be $x^3ya^{-2}b^{-3}c^{-4}$.

68. Strictly speaking, then, there is but one case in which the division of monomials will not give an entire quotient, and that is, when the coefficient of the dividend is not exactly divisible by the coefficient of the divisor.

CASE II.

69. When the dividend is a compound quantity, and the divisor a monomial.

The dividend may be regarded as the product of each term of the quotient sought by the monomial divisor; hence, to find this quotient, we must divide each term of the dividend by the divisor.

RULE.

70. *Divide each term of the dividend separately by each term of the divisor, as in Case I., and connect the partial quotients by their appropriate signs.*

EXAMPLES.

1. Divide $6x^2y^2 - 4x^3yz + 8az$ by $2z$.
 Ans. $3x^2y^2z^{-1} - 2x^3y + 4a$.
2. Divide $x^2 + 2ax + a^2$ by x. *Ans.* $x + 2a + a^2x^{-1}$.
3. Divide $x^p + x^{p+1} + x^{p+2} + x^{p+3}$ by x^p.
 Ans. $1 + x + x^2 + x^3$
4. Divide $x^{-p} + x^{-p+1} + x^{-p+2} + x^{-p+3}$ by x^{-p}.
 Ans. $1 + x + x^2 + x^3$.
5. Divide $x^{-p-1} - x^{-p-2} - x^{-p-3} - x^{-p-4}$ by x^p.
 Ans. $x^{-2p-1} - x^{-2p-2} - x^{-2p-3} - x^{-2p-4}$.
6. Divide $x^{-p-1} - x^{-p-2} - x^{-p-3} - x^{-p-4}$ by x^{-p}.
 Ans. $x^{-1} - x^{-2} - x^{-3} - x^{-4}$.
7. Divide $3x^2 + 2y + z^2$ by $2z$. *Ans.* $\frac{3}{2}x^2z^{-1} + yz^{-1} + \frac{z}{2}$.
8. Divide $3x^2 + 2y + z^2$ by $2z^{-1}$. *Ans.* $\frac{3}{2}x^2z + yz + \frac{z^3}{2}$.
9. Divide $40a^2b^2 - 10abx + 15abxy$ by $5ab$.
 Ans. $8ab - 2x + 3xy$.
10. Divide $40a^2b - 10abx + 15abxy$ by $-5ab$.
 Ans. $-8a + 2x - 3xy$.

CASE III.

71. When the dividend and divisor are both polynomials.

We must keep in view that the dividend is the collection, after addition and subtraction, of the partial products arising from multiplying each term of the quotient, when found, by each term of the divisor. If, then, we can find a true term of the quotient and multiply it into each term of the divisor, we will form so much of the dividend as was composed of the product of this term by the whole divisor. And when we have subtracted this product from the dividend, the remainder will be made up of the partial products of the remaining terms of the quotient not yet found by each term of the divisor. Now, if we can find another true term of the quotient and multiply it into the divisor, the product will be so much of the dividend as is made up by the multiplication of this second term of the quotient by each term of the divisor. And if we subtract this product from the first remainder, the new remainder, if there be any, will be so much of the dividend as is made up of the product of the remaining terms of the quotient not yet found by each term of the divisor. We can thus regard each remainder in succession as a new dividend until all the terms of the quotient are found.

72. The whole difficulty then consists in finding, in succession, true terms of the quotient. Now, we have seen that in the product of polynomials there are always two terms which are dissimilar from the other terms, and, consequently, irreducible with them. They are the terms arising from the multiplication of the two terms in the multiplicand and multiplier affected with the highest and lowest exponent of the same letter. If, then, we divide that term of the dividend which contains the highest exponent of a certain letter by that term of the divisor which contains the highest exponent of the same letter, we are sure of getting a true term of the quotient. For this term of the dividend has been formed without reduction from the multiplication of the corresponding term in the divisor by the quotient found.

73. In like manner, if we divide the terms affected with the lowest exponent of the same letter, the one by the other, we are sure of getting a true term of the quotient sought. For this reason, the dividend and divisor are arranged with reference to the highest or lowest exponent of the same letter, generally with reference to the highest. When so arranged, the first term of each successive remainder will contain the highest or lowest exponent of the letter according to which the polynomials are arranged, and will, when divided by the first term of the divisor, give a true term of the quotient.

The divisor in Algebra is written on the right of the dividend, and the quotient immediately under the divisor.

Divide $a^2 + 2ab + b^2$ by $a + b$.

Dividend.	*Divisor.*
$a^2 + 2ab + b^2$	$a + b$
$a^2 + ab$	$a + b$
$+ ab + b^2$	*Quotient.*
$ab + b^2$	
$0 + 0$	*Remainder.*

or

Dividend.	*Divisor.*
$b^2 + 2ba + a^2$	$b + a$
$b^2 + ba$	$b + a$
$ba + a^2$	*Quotient.*
$ba + a^2$	
$0 + 0$	*Remainder.*

74. We see, that by arranging dividend and divisor with reference to the ascending or descending powers of the same letter, and dividing the first term of the dividend and the first of the remainder by the first term of the divisor, we necessarily get true terms of the quotient. But if the polynomials are not arranged we will not get true terms of the quotient. Write the dividend and divisor thus,

Dividend.	*Divisor.*
$b^2 + 2ab + a^2$	$a + b$
$b^2 + \frac{b^3}{a}$	$\frac{b^2}{a}$ — &c., *Quotient.*
$-\frac{b^3}{a} + 2ab + a^2$	

We get $\frac{b^2}{a}$ for the first term of the quotient, which is not a true result. It is also plain that the division would never end.

75. Since each successive remainder may be regarded as a new polynomial, to be divided by the divisor, we can change the arrangement of the remainders at pleasure, provided we make a corresponding change in the divisor. In the example above we might arrange the remainder $ab + b^2$ with reference to b, and write it $b^2 + ab$, provided we change the arrangement of the divisor, and write it $b + a$.

RULE.

76. *Arrange the dividend and divisor with reference to the ascending or descending powers of the same letter, and then divide the first term on the left of the dividend by the first term on the left of the divisor. This will give the first term of the quotient.*

Multiply this term into each term of the divisor, and subtract the product from the dividend. Divide the first term of the remainder by the first term of the divisor for the second term of the quotient. Multiply this term into the divisor, as before, and subtract the product from the first remainder.

Continue this process, dividing the first term of each remainder by the first term of the divisor until we get a remainder, zero, when the division is said to be exact.

But, if the exponents of the letter, according to which the arrangement is made, are all positive, and the first term of any remainder is *not divisible by the first term of the divisor, the quotient is incomplete; or, if some of the exponents are negative, and the letter, according to which the arrangement has been made, has disappeared from any of the successive remainders, the quotient is also incomplete.*

EXAMPLES.

1. Divide $x^3 + 3ax^2 + 3a^2x + a^3$ by $x + a$.
   Ans. $x^2 + 2ax + a^2$.
2. Divide $x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + x^4$ by $x + a$.
   Ans. $x^3 + 3ax^2 + 3a^2x + a^3$.
3. Divide $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$.
   Ans. $x^2 - 2xy + y^2$.
4. Divide $4a^2 - b^2$ by $2a + b$.
   Ans. $2a - b$.
5. Divide $x^m - y^m$ by $x - y$.
   Ans. $x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \dfrac{x^{m-3}y^3 - y^m}{x - y}$
6. Divide $x^8 - y^8$ by $x^2 - y^2$.
   Ans. $x^6 + x^4y^2 + x^2y^4 + y^6$.
7. Divide $x^{12} - y^{12}$ by $x^2 - y^2$.
   Ans. $x^{10} + x^8y^2 + x^6y^4 + x^4y^6 + x^2y^8 + y^{10}$.
8. Divide $x^7 - y^7$ by $x^2 - y^2$.
   Ans. $x^5 + x^3y^2 + xy^4 + \dfrac{xy^6 - y^7}{x^2 - y^2}$.
9. Divide $x^9 - y^9$ by $x^3 - y^3$.
   Ans. $x^6 + x^3y^3 + y^6$.
10. Divide $x^8 + y^8$ by $x^2 + y^2$.
    Ans. $x^6 - x^4y^2 + x^2y^4 - y^6 + \dfrac{2y^8}{x^2 + y^2}$.
11. Divide $x^{12} + y^{12}$ by $x^2 + y^2$.
    Ans. $x^{10} - x^8y^2 + x^6y^4 - x^4y^6 + x^2y^8 - y^{10} + \dfrac{2y^{12}}{x^2 + y^2}$.
12. Divide $x^7 + y^7$ by $x^2 + y^2$.
    Ans. $x^5 - x^3y^2 + xy^4 - \dfrac{xy^6 + y^7}{x^2 + y^2}$.
13. Divide $x^9 + y^9$ by $x^3 + y^3$.
    Ans. $x^6 - x^3y^3 + y^6$.

14. Divide $x^{15} + y^{15}$ by $x^5 + y^5$.

Ans. $x^{10} - x^5y^5 + y^{10}$.

15. Divide $x^{15} + y^{15}$ by $x^3 + y^3$.

Ans. $x^{12} - x^9y^3 + x^6y^6 - x^3y^9 + y^{12}$.

16. Divide $7ax - 7ay + 7ab + bx - by + b^2$ by $7a + b$.

Ans. $x - y + b$.

17. Divide $x^2y^{-3} + x^{-1} + x^{-3}y^2 + y^{-1}$ by $x^2 + y^2$.

Ans. $x^{-3} + y^{-3}$.

18. Divide $14x^8 - 14y^8$ by $7x^2 - 7y^2$.

Ans. $2x^6 + 2x^4y^2 + 2x^2y^4 + 2y^6$.

19. Divide $a^2 - 2ax + x^2 + 2ay - 2xy$ by $2y + a - x$.

Ans. $a - x$.

20. Divide $ax^2-2arx+ar^2+bx^2-2brx+br^2+cr^2-2crx+cx^2$ by $x-r$.

Ans. $(x - r)(a + b + c)$.

21. Divide $s^2 - 2sx + x^2 + rs - 2xr + r^2 + rs$ by $s - x + r$.

Ans. $s - x + r$.

22. Divide $50x^9 + 50y^9$ by $25x^3 + 25y^3$.

Ans. $2x^6 - 2x^3y^3 + 2y^6$.

23. Divide $(a + b + c)(x - r)^2$ by $(a + b + c)(x - r)$.

Ans. $x - r$.

24. Divide $x^4y^{-3} + x^4b + x^2 + y + y^4b + x^{-2}y^4$ by $x^4 + y^4$.

Ans. $y^{-3} + b + x^{-2}$.

25. Divide $49x^6y^8 - 64x^2y^2$ by $7x^3y^4 + 8xy$.

Ans. $7x^3y^4 - 8xy$.

26. Divide $a^6 - b^6$ by $a^2 + b^2$.

Ans. $a^4 - a^2b^2 + b^4 - \dfrac{2b^6}{a^2 + b^2}$.

27. Divide $a^{10} - b^{10}$ by $a^2 + b^2$.

Ans. $a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8 - \dfrac{2b^{10}}{a^2 + b^2}$.

28. Divide $a^8 - b^8$ by $a^2 + b^2$.

Ans. $a^6 - a^4b^2 + a^2b^4 - b^6$.

29. Divide $a^{12} - b^{12}$ by $a^3 + b^3$.

Ans. $a^9 - a^6b^3 + a^3b^6 - b^9$.

30. Divide $a^{20} - b^{20}$ by $a^5 + b^5$.

Ans. $a^{15} - a^{10}b^5 + a^5b^{10} - b^{15}$.

31. Divide 4 by $1 + x$.

Ans. $4 - 4x + 4x^2 - 4x^3 + \dfrac{4x^4}{1 + x}$.

32. Divide 4 by $x + 1$.

Ans. $4x^{-1} - 4x^{-2} + 4x^{-3} - \frac{4x^{-4}}{x + 1}$.

33. Divide $m - nx + px^2 - yx^3$ by $1 + x$.

Ans. $m - (m + n)\,x + (m + n + p)\,x^2$ — other terms.

34. Divide $1 + x$ by $1 - x$.

Ans. $1 + 2x + 2x^2 + 2x^3 +$ &c.

35. Divide $1 - x$ by $1 + x$.

Ans. $1 - 2x + 2x^2 - 2x^3 +$ &c.

36. Divide $x^2 + ay + ax + 1 + xy$ by $x + a$.

Ans. $x + y + \frac{1}{x + a}$.

37. Divide $x^3 + x^2 + 5$ by $x^2 + 2$.

Ans. $x + 1 + \frac{3 - 2x}{x^2 + 2}$.

Remarks.

77. I. When the exponents of the arranged letter in dividend and divisor are all positive, the division will not be exact if the first term of any of the remainders does not contain the arranged letter to a higher power than it is contained in the divisor. When, therefore, we get a remainder in which the first term contains the arranged letter to a lower degree than it is contained in the divisor, we need proceed no further, for we can never arrive at an exact quotient.

If some of the exponents of the letter, according to which the polynomials are arranged, are negative, the last rule will not hold good, for the result of the division will be a term affected with a negative exponent, and there ought to be one or more such terms in the quotient. If, however, the letter, according to which the arrangement has been made, has disappeared from any of the successive remainders, we need proceed no further.

II. Examples 6, 7, and 9, show that the difference between the like powers of two quantities is divisible by the like powers of a lower degree of those quantities, when the common exponent of the dividend, divided by the common exponent of the divisor, gives an exact quotient. And they show that the number of terms in the quotient is expressed by the quotient of the exponents. Thus, in example 6 there are four terms in the quotient, the same as $\frac{8}{2}$, the quotient of the exponents. In example 9, the quotient of the exponents is three, and the division is exact, with three terms in the result.

III. Examples 13, 14, and 15, show that the sum of the like powers of two quantities is divisible by the sum of the like powers of the same quantities when the result of the division of the exponents is odd.

IV. Examples 28, 29, and 30, show that the difference between the like powers of two quantities is divisible by the sum of their like powers when the quotient of the exponents is even. In the last two cases, as in the first, the quotient of the exponents gives the number of terms in the result.

V. If all the exponents of the dividend and divisor are positive, we can tell, by a simple inspection, in many examples, whether the division is possible; when the extreme terms of the arranged polynomials are not divisible by each other, the quotient will not be exact. Take $x^2 + 2xy + y^2$, to be divided by $x^3 + y^2$; the division is impossible, because the exponents of the result must be positive; and x^2 by x^3 will give x^{-1}. It, of course, does not follow that the division will give a complete quotient when the extreme terms are divisible by each other. Take $x^2 + 4 + y^2$, to be divided by $x + y$; the extreme terms are divisible by each other, but the quotient of the polynomials is not exact.

PRINCIPLES IN DIVISION.

78. I. We will now show that the difference between the like powers of two quantities will be divisible by the difference of their like powers of a lower degree, whenever the quotient of the common exponent of the dividend, by the common exponent of the divisor, is exact. That is, that $a^m - b^m$ is divisible by $a^n - b^n$, when m is exactly divisible by n.

$$\begin{array}{l|l} a^m - b^m & a^n - b^n \\ \hline a^m - a^{m-n}b^n & a^{m-n}. \\ \hline a^{m-n}b^n - b^m = b^n(a^{m-n} - b^{m-n}). & \end{array}$$

Performing the division, we get a^{m-n} for a quotient, and $b^n(a^{m-n} - b^{m-n})$ for a remainder. If this remainder is divisible by $a^n - b^n$, the dividend will also be divisible by $a^n - b^n$; for, represent the dividend by D, the divisor by d, the quotient by q, and the remainder by R. Then $a^m - b^m$, or $D = qd + R$. Now, qd is plainly divisible by d, and if R be also divisible by d, the first member must also be divisible by d, otherwise we would have the sum of two entire quantities equal to a fraction. Hence, in general, if the remainder is divisible by the divisor, the dividend will also be divisible by it. Then, if we can prove that the remainder, $b^n(a^{m-n} - b^{m-n})$, can be divided by $a^n - b^n$, we can prove that the dividend can also be divided by $a^n - b^n$. But if the factor, $a^{m-n} - b^{m-n}$, is divisible by the divisor, b^n times that factor will also be divisible by the divisor; and the remainder giving an exact quotient, the dividend will also give an exact quotient. That is, $a^m - b^m$ will be divisible by $a^n - b^n$ whenever $a^{m-n} - b^{m-n}$ is divisible by $a^n - b^n$; or, in other words, if the difference of the like power of two quantities is divisible by the difference of the like power of the n^{th} degree, the difference of the like powers of a degree higher by n will also be divisible by the difference of the n^{th} degree. But we know that $a^n - b^n$ is divisible by itself, $a^n - b^n$; hence, by the principle just demonstrated, $a^{2n} - b^{2n}$ must be divisible by $a^n - b^n$. And since $a^{2n} - b^{2n}$ is divisible by $a^n - b^n$, $a^{3n} - b^{3n}$ must also be divisible by $a^n - b^n$, and so on, for powers of a degree greater by n, until it finally reaches and divides $a^m - b^m$. This power must eventually be reached, because m is supposed to be a multiple of n.

79. In Example 8, $x^7 - y^7$, divided by $x^2 - y^2$, the quotient was not exact, because m or 7 was not a multiple of n or 2.

In Examples 6, 7, and 9, the quotients were exact, because in each case m was a multiple of n.

80. It is plain that $pa^m - pb^m$ can be divided by $qa^n - qb^n$ when p and m are multiples of q and n. For we can put the expressions under the form of $p(a^m - b^m)$, and $q(a^n - b^n)$, and the quotient will, of course, be exact when p will divide q, and $a^m - b^m$ will divide $a^n - b^n$.

Thus, in Example 18 we found $14x^8 - 14y^8$ divisible by $7x^2 - 7y^2$: the dividend can be written, $14(x^8 - y^8)$, and the divisor $7(x^2 - y^2)$, and since 14 will divide 7, and $x^8 - y^8$ will divide $x^2 - y^2$, the quotient is exact.

81. The demonstration holds good whenever n, added to itself a certain number of times, will produce m, and is therefore true for quantities affected with negative and fractional exponents.

Divide $a^{-4} - b^{-4}$ by $a^{-2} - b^{-2}$. *Ans.* $a^{-2} + b^{-2}$.

Divide $a^{-6} - b^{-6}$ by $a^{-3} - b^{-3}$. *Ans.* $a^{-3} + b^{-3}$.

Divide $a^{-6} - b^{-6}$ by $a^{-2} - b^{-2}$. *Ans.* $a^{-4} + a^{-2}b^{-2} + b^{-4}$.

Divide $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$. *Ans.* $a^{\frac{1}{4}} + b^{\frac{1}{4}}$.

Divide $a^{\frac{3}{2}} - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$. *Ans.* $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.

Divide $a^{\frac{5}{6}} - b^{\frac{5}{6}}$ by $a^{\frac{1}{6}} - b^{\frac{1}{6}}$.

Ans. $a^{\frac{2}{3}} + a^{\frac{1}{2}}b^{\frac{1}{6}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + a^{\frac{1}{6}}b^{\frac{1}{2}} + b^{\frac{2}{3}}$.

82. II. The difference of the like powers of two quantities is divisible by the sum of their like powers of a lower degree, when the result of the division of the common exponent of the dividend by the common exponent of the divisor is an even number; which is evident from the principle of Article 50.

83. III. The sum of the like powers of two quantities is divisible by the sum of their like powers of a lower degree, when the quotient arising from dividing the common exponent of dividend by the common exponent of the divisor is an odd number.

That is, $a^m + b^m$ is divisible by $a^n + b^n$, when m, divided by n, is an odd number.

Performing the division, we will have $-b^n(a^{m-n} - b^{m-n})$ for a remainder, and whenever the factor, $a^{m-n} - b^{m-n}$, is divisible by $a^n + b^n$, the dividend $a^m + b^m$ will be divisible by the same divisor. But we have just shown that $a^{m-n} - b^{m-n}$ can be divided by $a^n + b^n$ when $\frac{m-n}{n}$ is an even number, and since $\frac{m-n}{n}$ is the same as $\frac{m}{n} - 1$, the quotient of $\frac{m}{n}$ must be odd in case the quotient of $\frac{m-n}{n}$ is even. Hence, the truth of the proposition. Thus, in Examples 13, 14, and 15, the quotients were exact, because for each of these examples $\frac{m}{n}$ was an odd number. So, in like manner, $x^3 + a^3$, $x^5 + a^5$, $x^7 + a^7$, &c., are divisible by $x + a$, and give 3, 5, 7, &c., terms in the quotient, with signs alternately plus and minus.

84. IV. It is plain that the sum of the like powers of two quantities cannot be divided by the difference of their like powers; that is, $a^m + b^m$ cannot be divided by $a^n - b^n$, because it is not possible to decompose the sum of two quantities into factors, one of which will be a difference between the quantities.

ALGEBRAIC FRACTIONS.

85. A fraction is a *broken* part of unity. The denominator denotes the number of equal parts into which the unit has been broken or divided, and the numerator expresses the number of these equal parts taken. Thus, the fraction $\frac{2}{3}$ indicates that the unit has been divided into three equal parts, and that two of these parts have been taken. In like manner, the fraction $\frac{a}{b}$ indicates that the unit has been divided into b equal parts, and that a of these parts have been taken.

86. Every quantity not expressed under a fractional form is called an entire quantity.

87. An expresssion, made up in part of an entire quantity, and in part of a fraction, is called a mixed quantity. Thus, $4 + \frac{1}{2}$, and $a - \frac{b}{c}$ are mixed quantities.

88. A proper fraction is one in which the numerator is less than the denominator. An improper fraction is one in which the numerator is greater than the denominator.

89. A simple fraction is one whose numerator and denominator are simple quantities. Thus, $\frac{a}{b}$ is a simple fraction.

90. A compound fraction is one which has a compound expression in the numerator or denominator, or in both. Thus, $\frac{a+b}{c}$, $\frac{a}{c+d}$ and $\frac{a+b}{c+d}$ are compound fractions.

91. The minus sign before a fraction changes the signs of all the terms in the numerator. Thus, $-\frac{+c-d}{b}$ is equivalent to $\frac{d-c}{b}$.

92. A few of the principles involved in operations upon fractions will now be demonstrated.

I. The multiplication of a fraction by an entire quantity is effected by multiplying the numerator, or dividing the denominator by the entire quantity.

For to multiply $\frac{a}{b}$ by c, is to repeat $\frac{a}{b}$, c times, and since $\frac{a}{b}$ taken twice, is $\frac{2a}{b}$, three times is $\frac{3a}{b}$, &c, the result of multiplying $\frac{a}{b}$, c times will plainly be $\frac{ac}{b}$. The multiplication can also be performed by dividing the denominator by c; for, if we divide the denominator by c, and q is the quotient of the division, then the unit will be divided into c times fewer parts than before, and, of course, each part is c times as great as before. Now, if we write a over q, we will have taken as many parts of the unit as before; and since each part is c times greater, the fraction $\frac{a}{q}$ is plainly c times greater than the fraction $\frac{a}{b}$.

The multiplication of the numerator by a whole number may also be demonstrated in another way. $\frac{a}{b}$ is taken to be increased c fold, and if the denominator remains unchanged, while the numerator is multiplied by c, the number of parts into which the unit is divided remains the same, but the number of those parts taken is increased c fold, the new fraction $\frac{ac}{b}$ must then be c fold greater than the old.

93. II. The division of a fraction by an entire quantity is effected by dividing the numerator, or multiplying the denominator by the entire quantity.

For, to divide $\frac{a}{b}$ by c, is to diminish the value of the fraction c fold, and if the result of the division of a by c is q; then q is c times smaller than a, and $\frac{q}{b}$ is c times smaller than $\frac{a}{b}$. Because, while the parts of the unit have remained unchanged, c fold fewer of these parts have been taken.

The division can also be effected by multiplying the denominator by c, for then the parts into which the unit is divided being increased c fold, the value of each part must be decreased c fold, and, of course, when the same number of these c fold diminished parts are taken, as at first, the result must be c times smaller than before.

94. III. The value of a fraction is not altered by multiplying the numerator and denominator by the same quantity. The fraction $\frac{a}{b}$ is not altered in value by multiplying the numerator and denominator by c. For, to multiply the numerator by c is, from what has just been

shown, to increase the value of the fraction c fold, and to multiply the denominator by c is to diminish the value of the fraction c fold, and, of course, the fraction has undergone no change of value since it has been increased and decreased equally.

95. IV. The value of a fraction is not changed by dividing the numerator and denominator by the same quantity. Because the two divisions cancel each other, and leave the fraction in its primitive condition.

96. V. If the same quantity be added to the numerator and denominator of a proper fraction, the value of the fraction will be increased.

To show this, it will be necessary to show that if the same quantity be added to two quantities differing in magnitude, the smaller of these will be increased more, proportionally, than the larger. Take the numbers 1 and 2, add 1 to both, the first will be doubled, but the second will not be; add 3 to both, the first will be increased four fold, while the second will only be $2\frac{1}{2}$ times greater than before. In general, let a and b represent the two quantities, a being less than b; add a to both, the first will be doubled, but the second will not be, because $a + b$ is less than $2b$. Now, when the same quantity is added to the numerator and denominator of a proper fraction, the numerator is increased more proportionally than the denominator; the number of parts taken, then, is increased without their being an equal decrease in the size of those parts. Hence, the new fraction must be greater than the old.

97. VI. By adding large quantities to the numerator and denominator, we can make the value of the fraction approximate indefinitely near, though it can never become unity. Thus, add 1000 to the numerator and denominator of $\frac{1}{2}$, the new fraction is $\frac{1001}{1002}$, almost, though not quite, unity. Now, add a million to both terms of the fraction, and the result will be indefinitely near, without being altogether unity.

98. VII. If the same quantity is added to the numerator and denominator of an improper fraction, the value of the fraction will be decreased.

For, the denominator being smaller than the numerator, will be increased more proportionally, and the value of the fraction must be diminished.

Take $\frac{5}{2}$ as an example, add 10, and the fraction becomes $\frac{15}{12} < \frac{5}{2}$. It is plain that no addition to the numerator and denominator can ever reduce the fraction as low as unity.

REDUCTION OF FRACTIONS.

99. The reduction of fractions consists in simplifying their forms without altering their value.

Their are 10 cases.

CASE I.

To reduce a simple fraction to its simplest form.

RULE.

Strike out all the factors common to the numerator and denominator, and the result will be the fraction reduced to its lowest terms.

Take as an example, $\frac{a^2 - b^2}{a + b}$

We know the factors of $a^2 - b^2$ to be $(a + b)(a - b)$, hence,

$$\frac{a^2 - b^2}{a + b} = \frac{(a - b)(a + b)}{a + b} = a - b.$$

The expression $a + b$, being common to numerator and denominator, may be stricken out, since, by Article 95, we can divide both terms of the fraction by $a + b$ without altering its value

Take $\frac{n^2 + 2n + 1}{n + 1}$.

We know, by Article 48, that $n^2 + 2n + 1 = (n + 1)^2$, hence,

$$\frac{n^2 + 2n + 1}{n + 1} = \frac{(n + 1)(n + 1)}{(n + 1)} = n + 1.$$

3. Reduce $\frac{x^2 - 2ax + a^2}{x - a}$. *Ans.* $x - a$.

4. Reduce $\frac{12a^2 - 12b^2}{4a - 4b}$. *Ans.* $3(a + b)$.

5. Reduce $\frac{6a^2xc + 6axm}{36ax + 42a^3x^3}$. *Ans.* $\frac{ac + m}{6 + 7a^2x^2}$.

6. Reduce $\frac{4m^2 - 4n^2}{8(m+n)}$ *Ans.* $\frac{m-n}{2}$.

7. Reduce $\frac{ac + bc + c^2 + dc}{a + b + c + d}$. *Ans.* c.

8. Reduce $\frac{3xy + 9x^2y^2 - 27x^3y^3c}{9bxy + 3axy + 12x^2y^2}$.

Ans. $\frac{1 + 3xy - 9x^2y^2c}{3b + a + 4xy}$.

CASE II.

100. To reduce a mixed quantity to the form of a fraction.

RULE.

Multiply the entire quantity by the denominator of the fraction; connect the product with the numerator by the appropriate sign of the numerator, and write the denominator under the whole result.

Take for an example, $a + \frac{x}{b}$, the result is $\frac{ab + x}{b}$. We have not altered the value of the fraction, since the division by b will again give us $a + \frac{x}{b}$. We have, in fact, only multiplied and divided a by the same quantity, b.

2. Reduce $a + \frac{x - a^2}{a}$ *Ans.* $\frac{x}{a}$.

3. Reduce $2x + \frac{x^2 - 2ax + a^2}{a}$. *Ans.* $\frac{x^2 + a^2}{a}$.

4. Reduce $m - n + \frac{m^2 - n^2}{m + n}$. *Ans.* $2\frac{(m^2 - n^2)}{m + n}$.

5. Reduce $a - 2x + \frac{a^2 - 2ax + x^2}{x - a.}$. *Ans.* $-x$.

6. Reduce $1 + x + \frac{2 - x^2}{1 - x}$. *Ans.* $\frac{3 - 2x^2}{1 - x}$.

7. Reduce $7 - a + \dfrac{x+b}{c}$. *Ans.* $\dfrac{7c - ac + x + b}{c}$.

8. Reduce $4 - \dfrac{b^2 + 4c^2 + x^2}{c^2 + x^2}$. *Ans.* $\dfrac{3x^2 - b^2}{c^2 + x^2}$.

9. Reduce $4x + 3a - \dfrac{x^2 - 2ax + 3a^2}{x+a}$.

Ans. $\dfrac{9ax + 3x^2}{x+a}$.

10. Reduce $x - a + \dfrac{2a^2}{x+a}$. *Ans.* $\dfrac{x^2 + a^2}{x+a}$.

11. Reduce $b + 2y + \dfrac{-b^2 - 2by - y^2}{b+y}$. *Ans.* y.

12. Reduce $11x + 11y - 11 + \dfrac{11y^2 - 11x^2 + 11x - 33y + 22}{x - y + 1}$.

Ans. 11.

CASE III.

101. To reduce a fraction to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator for the entire part, and place the remainder, if any, over the denominator for the fractional part.

Take $$\frac{x^2 + a^2}{x + a}.$$

We have a right to divide both terms of the fraction (Article 95) by the denominator, and we will not alter the value of the fraction. The reduction in Case III, is nothing more than dividing both terms of the fraction by the denominator, and, since the new denominator is unity, it need not be written.

Hence, $$\frac{x^2 + a^2}{x + a} = \frac{x^2 + a^2}{x + a} \div \frac{x + a}{x + a},$$

that is, both terms divided by $x + a$.

The result is $\dfrac{x + \dfrac{-ax + a^2}{x + a}}{1}$, or simply, $x + \dfrac{-ax + a^2}{x + a}$.

1. Reduce $\dfrac{a^2 + k^2}{a}$. *Ans.* $a + \dfrac{k^2}{a}$.

2. Reduce $\dfrac{x^2 - 2ax + a^2 + 1}{x - a}$. *Ans.* $x - a + \dfrac{1}{x - a}$.

3. Reduce $\dfrac{2(m^2 - n^2)}{m + n}$. *Ans.* $2(m - n)$.

4. Reduce $\dfrac{1 + x}{1 - x}$. *Ans.* $1 + \dfrac{2x}{1 - x}$.

5. Reduce $\dfrac{1 - x}{1 + x}$. *Ans.* $1 - \dfrac{2x}{1 + x}$.

6. Reduce $\dfrac{1}{1 + x}$. *Ans.* $1 - \dfrac{x}{1 + x}$

7. Reduce $\dfrac{1}{1 - x}$. *Ans.* $1 + \dfrac{x}{1 - x}$.

8. Reduce $\dfrac{7a^2 + 7b^2}{a^2 + b^2}$. *Ans* 7.

9. Reduce $\dfrac{7a^2 + b^2}{a^2 + b^2}$. *Ans.* $7 - \dfrac{6b^2}{a^2 + b^2}$.

10. Reduce $\dfrac{a^3 - b^3}{a - b}$. *Ans.* $a^2 + ab + b^2$.

CASE IV.

102. To develop a fraction into a series.

RULE.

Divide the numerator by the denominator, as above, and continue the division as far as may be required, or may be deemed necessary.

This case differs from the last in two particulars, 1st, It includes only fractions which cannot be reduced to entire quantities. 2d. The division, instead of stopping with one term of the quotient, is carried on as far as may be thought proper.

EXAMPLES.

1. Expand $\frac{1+x}{1-x}$ into a series.

Ans. $1 + 2x + 2x^2 + 2x^3 +$ &c., indefinitely.

2. Expand $\frac{1-x}{1+x}$ into a series of five terms.

Ans. $1 - 2x + 2x^2 - 2x^3 + 2x^4 -$ &c.

3. Expand $\frac{1}{1+x}$ into a series of six terms.

Ans. $1 - x + x^2 - x^3 + x^4 - x^5 +$ &c

4. Expand $\frac{1}{1-x}$ into a series.

Ans. $1 + x + x^2 + x^3 + x^4 +$ &c.

5. Expand $\frac{7a^2 + b^2}{a^2 + b^2}$ into a series.

Ans. $7 - \frac{6b^2}{a^2} + \frac{6b^4}{a^4} - \frac{6b^6}{a^6} +$ &c.

6. Expand $\frac{1}{-x+1}$ into a series.

Ans. $-x^{-1} - x^{-2} - x^{-3} - x^{-4} - x^{-5} -$ &c.

7. Expand $\frac{1}{x+1}$ into a series of five terms.

Ans. $x^{-1} - x^{-2} + x^{-3} - x^{-4} + x^{-5} -$ &c.

8. Expand $\frac{x^2}{x - 2x^2 + x^3}$ into a series.

Ans. $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 +$ &c.

9. Expand $\frac{x^2 + y^2}{x + y}$.

Ans. $x - y + \frac{2y^2}{x} - \frac{2y^3}{x^2} + \frac{2y^4}{x^3} - \frac{2y^5}{x^4} +$ &c.

10. Expand $\frac{4}{x+y}$.

Ans. $4x^{-1} - 4x^{-2}y + 4x^{-3}y^2 - 4x^{-4}y^3 + \&c.$

Remarks.

103. In all these examples, since the quotient is incomplete, it is plain that the product of the divisor by the series will not give the dividend.

104. The different results of the same division in 4 and 6, 3 and 7, show that the series may be changed by changing the order of the terms in the divisor; and it might readily be shown that a change in the order of the terms in the numerator (when there is more than one term) will produce a corresponding change in the series.

CASE V.

105. To reduce fractions having different denominators to equivalent fractions having the same denominator.

RULE.

Multiply each numerator into all the denominators, except its own, for a new numerator, and all the denominators together for a common denominator of all the new numerators; or, multiply each numerator by the quotient arising from the division of the least common multiple of the denominators by each denominator respectively, and write the products thus formed for new numerators over the least common multiple as a common denominator.

Reduce $\frac{x}{a}$ and $\frac{b}{c}$ to a common denominator.

By the first rule we get $\frac{cx}{ac}$ and $\frac{ab}{ac}$, and it is plain that these fractions have the same value as at first, since each has been multiplied and divided by the same quantity.

Reduce $\frac{x}{a^2}$ and $\frac{b}{a}$ to a common denominator.

The least common multiple is a^2; the quotient of this common multiple by the first denominator is 1, and this is, then, the multiplier of the first numerator. The multiplier of the second numerator is $\frac{a^2}{a} = a$. Hence, the equivalent fractions are $\frac{x}{a^2}$ and $\frac{ab}{a^2}$. It is plain that each fraction has been multiplied and divided by the same quantity.

3. Reduce $\frac{x}{a^3}$, $\frac{y}{a^2}$, and $\frac{b}{a}$ to equivalent fractions with a common denominator.

Ans. $\frac{x}{a^3}, \frac{ay}{a^3}, \frac{a^2b}{a^3}$.

4. Reduce $\frac{1}{2}$, a, $\frac{x^2}{4}$, and $\frac{y}{b}$ to equivalent fractions having a common denominator.

Ans. $\frac{2b}{4b}, \frac{4ab}{4b}, \frac{bx^2}{4b}$, and $\frac{4y}{4b}$.

5. Reduce $\frac{1}{2}$, $\frac{m^2 - n^2}{m + n}$, $\frac{x^2 + a^2}{(m + n)^2}$, and $\frac{b}{4}$ to equivalent fractions with a common denominator.

Ans. $\frac{2\,(m + n)^2}{4\,(m + n)^2}, \frac{4\,(m + n)^2\,(m - n)}{4\,(m + n)^2}, \frac{4\,(x^2 + a^2)}{4\,(m + n)^2}$, and $\frac{(m + n)^2 b}{4\,(m + n)^2}$.

6. Reduce $\frac{a}{x}$, $\frac{x + a}{a}$, $\frac{ax}{a^2}$ to equivalent fractions having a common denominator.

Ans. $\frac{a^3}{a^2x}, \frac{ax^2 + a^2x}{a^2x}, \frac{ax^2}{a^2x}$.

7. Reduce $\frac{x}{m - n}$, $\frac{c}{b}$, $\frac{m - n}{x}$, $\frac{b}{c}$ to equivalent fractions having a common denominator.

Ans. $\frac{bcx^2}{(m - n)\,bcx}, \frac{(m - n)\,c^2x}{(m - n)\,bcx}, \frac{(m - n)^2bc}{(m - n)\,bcx}, \frac{(m - n)\,b^2x}{(m - n)\,bcx}$.

By reducing the above results by Case I., we will get back the original fractions. We can thus verify the correctness of the results obtained.

CASE VI.

106. To add fractional quantities together.

RULE.

Reduce the fractions to a common denominator, if necessary, and over this common denominator write the algebraic sum of the new numerators.

Fractions, which have different denominators, cannot be added together previous to reduction to the same denominator, because they represent different things. Thus, $\frac{1}{3}$ and $\frac{1}{4}$ neither make $\frac{2}{3}$ nor $\frac{2}{4}$; the first indicates that one of the three parts into which the unit has been been divided has been taken; the second indicates that we have taken one of the four parts into which unity has been divided.

Since, therefore, the parts taken are different in magnitude, they cannot be added together. It would, obviously, be just as proper to add a peck (the fourth of a bushel) and a quart (the thirty-second part of a bushel), and call the sum two pecks, or two quarts, as to add two fractions with different denominators.

EXAMPLES.

1. Add $\frac{x}{a} + \frac{b}{a}$, and $\frac{-c}{a}$ together. *Ans.* $\frac{x+b-c}{a}$.

2. Add $\frac{x}{a^2} + \frac{b}{a}$ and $-\frac{c}{a}$. *Ans.* $\frac{x+ba-ac}{a^2}$.

3. Add $\frac{4+x}{2}$ to $\frac{-b}{c}$. *Ans.* $\frac{4c+cx-2b}{2c}$.

4. Add $x-a+\frac{x^2+2ax-a^2}{2x-2a}$. *Ans.* $\frac{3x^2-2ax+a^2}{2(x-a)}$.

5. Add $\frac{2x}{3}+\frac{3x}{4}-\frac{4x}{5}+\frac{6x}{8}$. *Ans.* $\frac{164x}{120}$.

6. Add $\frac{x-y}{x}+\frac{b}{x+y}$ to $\frac{x+y}{c}-\frac{m}{n}$.

Ans. $\frac{nc(x^2-y^2)+bcxn+nx(x+y)^2-mcx(x+y)}{ncx(x+y)}$.

7. Add $\frac{x+y}{x-y}+\frac{c}{m}$ to $\frac{x-y}{x+y}+\frac{m}{c}$.

Ans. $\frac{2mc\ (x^2+y^2)+(c^2+m^2)\ (x^2-y^2)}{mc\ (x^2-y^2)}$.

8. Add $4+4x$ to $\frac{1}{4}+\frac{x}{4}$. *Ans.* $\frac{17\ (x+1)}{4}$.

9. Add $4-4x$ to $\frac{1}{4}-\frac{x}{4}$. *Ans.* $\frac{17\ (1-x)}{4}$.

10. Add $4x-4$ to $\frac{x}{4}-\frac{1}{4}$. *Ans.* $\frac{17\ (x-1)}{4}$.

11. Add $\frac{x+a}{b}-\frac{b}{x+a}$ to $x-a+\frac{b}{x-a}$.

Ans. $\frac{(x+a+bx-ba)\ (x^2-a^2)+2ab^2}{b\ (x^2-a^2)}$.

Remark.

107. When none of the terms have been reduced with each other, the result divided by the common denominator ought to give back the original series of fractions. This may be noticed in the first three examples.

CASE VII.

108. To subtract one or more fractional quantities from one or more fractional quantities.

RULE.

Make all the denominators the same, if the fractions have not a common denominator, then write into one sum the numerators of the minuend, and from this sum take the algebraic sum of the numerator, or numerators of the subtrahend.

Write the difference over the common denominator.

The same reasons which show that we cannot add fractions, with different denominators, prove that we cannot subtract fractions with unlike denominators. The fractions represent different things until reduced to a common denominator.

EXAMPLES.

1. Subtract $\frac{x}{2}$ from $\frac{x}{3}$. *Ans.* $-\frac{x}{6}$.

2. Subtract $\frac{x}{3}$ from $\frac{x}{2}$. *AnAns.* $\frac{x}{6}$.

3. Required the difference between x and $\frac{3x}{2}$.

Ans. $\frac{x}{2}$ or $-\frac{x}{2}$.

4. Take the difference between $x + \frac{a}{2}$ and $x - \frac{a}{2}$.

Ans. $+a$, or $-a$.

5. Subtract $\frac{a+b}{2} + \frac{x}{2}$ from $\frac{a-b}{2} - \frac{x}{2}$

Ans. $-2\,\frac{(b+x)}{2}$, or $-(b+x)$.

6. Subtract $\frac{a+b}{2} + x$ from $\frac{a-b}{2} - x$.

Ans. $-2\,\frac{(b+2x)}{2}$, or $-(b+2x)$.

7. Subtract $2x + \frac{a}{3} + 4$ from $\frac{y+b}{2}$.

Ans. $\frac{3(y+b) - 12x - 2a - 24}{6}$.

8. Subtract $\frac{y+b}{2}$ from $2x + \frac{a}{3} + 4$.

Ans. $\frac{12x + 2a + 24 - 3(y+b)}{6}$.

9. Subtract $\frac{a+x}{2}$ from $\frac{a-x}{3}$. *Ans.* $-\frac{(a+5x)}{6}$.

10. Subtract $\frac{a-x}{3}$ from $\frac{a+x}{2}$. *Ans.* $\frac{a+5x}{6}$.

11. Subtract $x - \frac{x}{b}$ from $\frac{x+c}{m}$.

Ans. $b\,\frac{(x+c) - mbx + mx}{mb}$.

CASE VIII.

109. To multiply fractional quantities together.

RULE.

If the quantities are mixed, reduce them to a fractional form, then multiply their numerators together for the numerator of the product required, and their denominators together for the denominator of this product.

Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

To multiply $\frac{a}{b}$ by c, is to repeat $\frac{a}{b}$, c times; the result will then plainly be $\frac{ac}{b}$. Because, while the size of the parts into which the unit has been divided has remained the same, the number of parts taken has been increased c fold. $\frac{ac}{b}$ is then, obviously, c times greater than $\frac{a}{b}$. But we were not required to multiply $\frac{a}{b}$ by c, but by the quotient arising from the division of c by d; our multiplier has then been d times too great, and, of course, the product is d times too great. The result, then, must be corrected by dividing by d. Hence, the true product of $\frac{a}{b}$ by $\frac{c}{d}$ is $\frac{ac}{bd}$, in accordance with the rule.

EXAMPLES.

1. Multiply $2x + \frac{b}{2}$ by $\frac{x}{2}$. *Ans.* $\frac{4x^2 + bx}{4}$.

2. Multiply $\frac{a + x}{2}$ by $\frac{a - x}{4}$. *Ans.* $\frac{a^2 - x^2}{8}$.

3. Multiply $\frac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{c}$ by $\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{a}$. *Ans.* $\frac{x - y}{ac}$.

4. Multiply $\frac{x^{-2} + y^{-2}}{x + y}$ by $\frac{x^{-2} - y^{-2}}{x - y}$. *Ans.* $\frac{x^{-4} - y^{-4}}{x^2 - y^2}$.

5. Multiply $\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$ by $\frac{x^{-1}-y^{-1}}{a+b}$.

Ans. $\frac{x^{-3}+x^{-1}y^{-2}-x^{-2}y^{-1}-y^{-3}}{(a+b)(x^{-1}+y^{-1})}$.

6. Multiply $\frac{x^2+y^3}{a}$ by $\frac{a}{x^2+y^3}$. *Ans.* 1.

7. Multiply $7+\frac{7y}{7-y}$ by $\frac{49-y^2}{7y+y^2}$. *Ans.* $\frac{49}{y}$.

8. Multiply $c+\frac{cx}{y}$ by $\frac{1}{c}+\frac{y}{cx}$. *Ans.* $\frac{(x+y)^2}{xy}$.

9. Multiply $\frac{1+4x-y}{a+b}$ by $\frac{1-4x+y}{ac-bc}$.

Ans. $\frac{1-16x^2+8xy-y^2}{c(a^2-b^2)}$.

10. Multiply $\frac{a+b}{1+4x-y}$ by $\frac{ac-bc}{1-4x+y}$.

Ans. $\frac{c(a^2-b^2)}{1-16x^2+8xy-y^2}$.

11. Multiply $8+\frac{a}{2}-c$ by $x-\frac{a}{2}+\frac{c}{4}$.

Ans. $\frac{64x+4ax-8cx-32a-2a^2+5ac+16c-2c^2}{8}$.

12. Multiply $\frac{x}{2}-\frac{b}{3}$ by $\frac{x}{2}+\frac{b}{3}$.

Ans. $\frac{9x^2-4b^2}{36}$ or $\frac{x^2}{4}-\frac{b^2}{9}$.

CASE IX.

110. To divide fractional quantities by each other.

RULE.

If the quantity to be divided is not in the form of a simple fraction, reduce it to that form; and if the quantity to be used as a divisor is not already a simple fraction, make it so. Then invert the terms of the divisor and proceed as in the last case.

Take, as an example, $\frac{a}{b}$ to be divided by $\frac{c}{d}$.

The result of the division of $\frac{a}{b}$ by c must be c times smaller than $\frac{a}{b}$. It will then plainly be $\frac{a}{bc}$, because, while the number of parts taken has remained unaltered, the size of these parts has been diminished c fold, since their number has been increased c fold. But we were not required to divide $\frac{a}{b}$ by c, but by the quotient of c by d. Our divisor, then, has been d times too large, and the result, $\frac{a}{bc}$, of course, d times too small. The error, then, must be corrected by multiplying by d.

Hence, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, in accordance with the rule.

111. The demonstration is general, and is applicable when the dividend and divisor are mixed quantities, or made up of several fractions; or, when either dividend or divisor is a mixed quantity, or composed of more than one fraction. For the fractions, if not already under the form of $\frac{a}{b}$ and $\frac{c}{d}$, can be put under these forms without difficulty.

The demonstration in the last case is general, for the same reasons.

EXAMPLES.

1. Divide $\frac{x^2}{2}$ by $\frac{x}{3}$. *Ans.* $\frac{3}{2}x$.

2. Divide $\frac{4x}{5}$ by 5. *Ans.* $\frac{4x}{25}$.

3. Divide $\frac{4x}{5}$ by $\frac{1}{5}$. *Ans.* $4x$.

4. Divide $\frac{x^2-1}{4}$ by $\frac{x+1}{8}$. *Ans.* $2(x-1)$.

5. Divide $\frac{x^2+2x+1}{x^2-1}$ by $\frac{x+1}{x-1}$. *Ans.* 1.

6. Divide $\frac{xy-y^2}{a+b}$ by $\frac{y}{a^2-b^2}$. *Ans.* $(x-y)(a-b)$.

7. Divide m by $\frac{(x-y)^2}{y} - y$. *Ans.* $\frac{my}{x^2 - 2xy}$.

8. Divide $\frac{my}{n}$ by $\frac{(x-y)^2}{y} + 2x$. *Ans.* $\frac{my^2}{n\,(x^2 + y^2)}$.

9. Divide $1 + \frac{x^2}{3} - 4x$ by $-1 - \frac{x^2}{3} + 4x$. *Ans.* -1.

10. Divide $1 + \frac{x^2}{3} - 4x$ by $+1 - \frac{x^2}{3} + 4x$.

Ans. $\frac{3 + x^2 - 12x}{3 - x^2 + 12x}$.

11. Divide $\frac{2\,(x+y)+ab}{a} - b + x$ by $\frac{(x-y)\,a + a^2x}{4\,(x+y) + 2ax}$.

Ans. $2\,\frac{[ax + 2\,(x+y)]^2}{a^2[x - y + ax]}$.

12. Divide $\frac{2\,(x+y)+ab}{a} - b + x$ by $\frac{4\,(x+y)+2ax}{(x-y)\,a + a^2x}$.

Ans. $\frac{x - y + ax}{2}$.

13. Divide $\frac{2a + x + 2}{b} - 2c + \frac{x}{b}$ by $\frac{m+n}{2b^2} - \frac{x}{b}$.

Ans. $\frac{4b\,(a + x - cb + 1)}{m + n - 2bx}$.

Remarks.

112. The quotient, multiplied by the divisor, ought to give the dividend; and in the last case, the product, divided by one of the factors, ought to give the other factor. We are thus enabled to verify our results.

We have seen that both the numerator and denominator of a fraction can be multiplied by the same quantity without altering the value of the fraction. Hence, by multiplying the term of a fraction by minus unity, the signs of these terms may be altered. Thus, $\frac{x-1}{b}$ may be written $\frac{1-x}{-b}$; $\frac{+a}{-b}$ may be written $\frac{-a}{+b}$, &c. Hence, $\frac{-a}{-b} = \frac{a}{b}$. That is, the quotient of two negative quantities is positive.

So, $\frac{-a}{+b}$, or, $\frac{+a}{-b} = -\frac{a}{b}$, read minus the fraction $\frac{a}{b}$.

CASE X.

113. To reduce a compound fraction to its lowest terms.

We have seen that a simple fraction can often be reduced by removing the factors common to the numerator and denominator. When the common factors of a compound fraction can be detected by inspection, we have but to remove them, and the fraction is reduced to its lowest terms. Thus,

$$\frac{a^2 + ab - ac}{am + an}.$$

can be reduced to its lowest terms by the removal of the common factor a.

But the common factor, or common factors, of a compound fraction cannot always be detected by inspection, and some process becomes necessary to discover them. This process is called finding the *greatest common* divisor.

114. The greatest quantity that will divide two or more quantities, is their greatest common divisor.

When the greatest common divisor of the numerator and denominator of a compound fractions is found, it can be reduced to its lowest terms by dividing by this divisor.

115. The greatest common divisor, though most usually obtained in order to reduce compound fractions, is also frequently found between quantities not written in the fractional form.

116. We will then explain the method of finding the greatest common divisor between two polynomials, without regarding them as numerator and denominator of a compound fraction.

117. The determination of the greatest common divisor of two polynomials depends upon two principles. 1st. The common divisor of two polynomials contains, as factors, all the common factors of the two polynomials, and does not contain any other factors. For, let A and B be the two polynomials, and D their greatest common divisor. Now, it is plain that any quantity C, made up of a part of the common factors of A and B, would divide both quantities, and would, therefore, be a common divisor; but C, multiplied by the remaining factors common to A and B, would still divide them. Hence, C would be a divisor, but not the greatest divisor of A and B.

Again, any quantity M, made up of all, or a part, of the common factors of A and B, and containing also a factor, d, not common to the two polynomials, will not divide them. For, if we proceed to the division, the common factors of A, B, and M will strike out, and leave the factor d as a denominator of the reduced polynomials. Thus, ad will not divide $a^2 + ab$ and $ac + am$; for, though it contains their common factor, a, it also contains a factor d, not common. The result of the division will be $\frac{a+b}{d}$ and $\frac{c+m}{d}$. The greatest common divisor, D, then, contains all the common factors of A and B, and contains no other factors.

118. 2d. The greatest common divisor of two polynomials, A and B, will enter into the successive remainders which arise from dividing A by B, and B by the remainder, and the second remainder by the first, and so on, until there is no remainder.

For, denote by A′ and B′ the quotients arising from dividing A and B by the greatest common divisor, D; then $A = A'D$, and $B = B'D$. Divide A by B; or, what is the same thing, divide A′D by B′D, and call the quotient Q.

Thus,

$$\begin{array}{l|l} A'D & B'D \\ \hline QB'D & Q \\ \hline (A' - QB')\,D = \text{1st Remainder} = MD, & \end{array}$$

by making $A' - QB' = M$.

We see that the first remainder contains the greatest common divisor; and as it is also in the divisor, we have a right to seek it between these polynomials.

Now, divide B′D by the remainder, and call the new quotient Q′, and we get

$$\begin{array}{l|l} B'D & MD \\ \hline Q'MD & Q' \\ \hline (B' - Q'M)\,D = \text{2d Remainder} = ND, & \end{array}$$

representing $B' - Q'M$ by it.

We see that the greatest common divisor enters also into the second remainder; and as it is also in the first remainder, we have a right to seek it between these two remainders. Divide MD by ND; the remainder, if any, will still contain the greatest common divisor.

119. Let us apply these principles in finding the greatest common divisor of the polynomials, A and B.

First arrange them with respect to a certain letter, and regard that polynomial which contains the highest power of this letter as the dividend, and the other polynomial as the divisor. Next, examine if A (which we suppose the dividend) contains a factor common to all its terms, but not common to all the terms of B. By the first principle, this factor can constitute no part of the common divisor, and may be suppressed. It is not absolutely necessary to suppress it, because it will disappear in division, and not appear in the remainder, and, therefore, by the second principle, cannot make part of the greatest common divisor. But if there is a factor common to all the terms of B, and not common to those of A, it must be suppressed. For A would not be divisible by B until it had been multiplied by the common factor of B, and the multiplication would make this factor common to A and B; and, hence, by the first principle, it must be common to the divisor. The greatest common divisor would then contain a factor which did not originally belong to A.

120. If there is a factor common to A and B which can be detected by inspection, it may be divided out and set aside, as making part of the greatest common divisor.

121. The next step, after setting aside, or suppressing factors seen by inspection, is to divide A by B. If the coefficient of the first term of A is not divisible by the coefficient of the first term of B, it may be made divisible by multiplying all the terms of A by the coefficient of the first term of B, or by any other quantity that will make the division possible. This multiplication will not effect the result, because the factor introduced into it will disappear in the division.

122. The remainder, after division of A by B, will contain the greatest common divisor; and, as B also contains it by hypothesis, we have a right to seek it between B and the remainder. The second remainder, if any, contains D also, but its coefficient or multiplier is smaller than in the first remainder; and so the coefficient of D, in each remainder, is smaller than in the preceding, until it finally becomes unity. When this final remainder is used as a divisor, it will go as many times in the preceding divisor as there are units in the coefficient of D in that divisor, and there will be no remainder.

123. When, therefore, we get a remainder zero, we conclude that the last divisor is $D \times 1$, or D itself.

124. If the given polynomials have no common divisor, we can discover the fact by the same tests that show when one polynomial is not divisible by another. We will find either that the letter, according to which the arrangement has been made, has disappeared from some of

the remainders, or it enters to a higher power in that remainder which is used as a divisor than in the one used as a dividend.

The preceding principles and demonstrations for finding the greatest common divisor lead to the following

RULE.

125. *Arrange the two polynomials with reference to a certain letter, as in division; use that one which contains the highest power of this letter as the dividend.*

2d. *Next, set aside the factors, if any, common to dividend and divisor, as part of the greatest common divisor, and suppress those factors which are common to the one and not to the other.*

3d. *Prepare the dividend, if necessary, for division, by multiplying by any quantity that will make its first term divisible by the first term of the divisor; divide the one polynomial by the other, and suppress in the remainder any factor that may be common to all its terms and not common to those of the divisor.*

4th. *Use the remainder so reduced as a divisor, and the last divisor as a dividend, and proceed as before; and continue in this manner, using each successive remainder as a divisor, and the last divisor as a dividend, until there is no remainder, or until it is evident that the polynomials have no common divisor.*

Reduce $\dfrac{a^2cb - a^2cy + a^2bx - a^2yx}{mcb + mcb + mbx + myx}$ to its lowest terms.

a^2 is common to the numerator and not to the denominator. m is common to the denominator and not to the numerator. These factors must then be suppressed, as constituting no part of the greatest common divisor.

Then,

$$\begin{array}{l|l}
cb - cy + bx - yx & cb + cy + bx + yx \\
\cline{2-2}
\underline{cb + cy + bx + yx} & \quad 1 \quad \text{Quotient.} \\
-2cy - 2yx = -2y(c + x). &
\end{array}$$

Suppress $-2y$, a factor common to the remainder and not to the divisor.

Then,

$$\begin{array}{l|l}
cb + cy + bx + yx & c + x \\
\cline{2-2}
\underline{cb + bx} & b + y \quad \text{2d Quotient.} \\
\quad cy + yx & \\
\quad \underline{cy + yx} & \\
\quad 0 + 0. &
\end{array}$$

Hence, $c + x$ is the greatest common divisor, and the reduced fraction obtained by dividing both terms of the fraction by the G, C, D, is

$$\frac{a^2(b-y)}{m(b+y)}.$$

The factor suppressed in the first remainder might have been $+2y$, instead of $-2y$. The greatest common divisor then would have been $-c-x$. And, in general, the two polynomials have two greatest common divisors, differing in the signs of all their terms. The reason of this is obvious, since $\frac{Ad}{Bd} = \frac{-Ad}{-Bd}$, the common divisor to the two terms of the fraction, may be either $+d$, or $-d$.

2. Find the greatest common divisor of $x^3 + 4x^2 + 5x + 2$, and $2x^2 + 3x + 1$. Prepare for division by multiplying the first polynomial by 4, the square of the coefficient of the first term of the second polynomial.

Then,

$$\begin{array}{l|l}
4x^3 + 16x^2 + 20x + 8 & 2x^2 + 3x + 1 \\
\underline{4x^3 + 6x^2 + 2x} & \overline{2x + 5} \quad \text{Quotient.} \\
10x^2 + 18x + 8 & \\
\underline{10x^2 + 15x + 5} & \\
3x + 3 = 3(x+1). & \text{Remainder.}
\end{array}$$

Suppress the common factor, 3, of the remainder, and continue the division.

$$\begin{array}{l|l}
2x^2 + 3x + 1 & x + 1 \\
\underline{2x^2 + 2x} & \overline{2x + 1.} \quad \text{2d Quotient.} \quad x + 1 = \text{G.C.D.} \\
x + 1 & \\
x + 1 &
\end{array}$$

126. If we had multiplied, to prepare for division, by the first instead of the second power of the coefficient of the first term, we would only have found one term in the quotient by the first division, and it would have been necessary to multiply the remainder by the coefficient of the first term to obtain a second term of the quotient. It is easier to prepare the polynomial by multiplying by the second power of the coefficient of the first term.

3. Reduce $\frac{x^4 + 4x^3 - 3x^2 - 10x + 8}{3x^2 - 4x + 1}$ to its lowest terms.

Multiply the numerator by $3^3 = 27$, to prepare for division.

$$\begin{array}{l|l}
27x^4 + 108x^3 - 81x^2 - 270x + 216 & 3x^2 - 4x + 1 \\
\hline
27x^4 - 36x^3 + 9x^2 & 9x^2 + 48x + 34 \quad \text{Quotient.} \\
\hline
\qquad 144x^3 - 90x^2 - 270x + 216 & \\
\qquad 144x^3 - 192x^2 + 48x & \\
\hline
\qquad\qquad 102x^2 - 318x + 216 & \\
\qquad\qquad 102x^2 - 136x + 34 & \\
\hline
\qquad\qquad\quad -182x + 182 = -182\,(x-1). &
\end{array}$$

Suppress -182.

Then,

$$\begin{array}{l|l}
3x^2 - 4x + 1 & x - 1 \\
\hline
3x^2 - 3x & 3x - 1 \\
\hline
\quad - x + 1 & \\
\quad - x + 1 & \\
\hline
\quad\ 0 \ . \ 0 &
\end{array}$$

Hence, $x - 1$ is the divisor sought, and the reduced fraction

$$\frac{x^3 + 5x^2 + 2x - 8}{3x - 1}.$$

127. In the above examples, the difference of the exponents of the arranged letter being two, we multiply by the square of the coefficient of the first term of the divisor to prepare the dividend for division. It could have been prepared by multiplying by 3, but we would only have gotten one term in the quotient, and there would have been two remainders to prepare for division.

In general, when preparation for division is necessary, we save time by multiplying the dividend by the coefficient of the first term of the divisor, raised to a power one greater than the difference of exponents of the arranged letter in the first terms of the two polynomials.

4. Find the greatest common divisor of

$$a^4 - a^3x + ax^2 - x^3, \text{ and } a^3 - a^2x + ax^2 - x^3.$$

Ans. $a - x$.

5. Reduce $\dfrac{cx^5 + cx^4 + cy^2 - x^6 - x^5 - xy^2}{cx^3 - cx^2 + cy^2 - x^4 + x^3 - xy^2}$.

Ans. GCD $c - x$.

And reduced fraction, $\dfrac{x^5 + x^4 + y^2}{x^3 - x^2 + y^2}$.

6. Reduce $\frac{cx^4 - cy^4 + yx^4 - y^5}{cx^2 - cy^2 + 2x^2 - 2y^2}$ to its lowest terms.

Ans. GCD $x^2 - y^2$.

Reduced fraction, $\frac{cx^2 + cy^2 + yx^2 + y^3}{c + 2}$.

7. Reduce $\frac{ma^2 - 2amx + mx^2 + a^2 - 2ax + x^2}{na^2 - 2anx + nx^2}$ to its lowest terms.

Ans. GCD $a^2 - 2ax + x^2$.

Reduced fraction, $\frac{m + 1}{n}$.

8. Find the GCD of $3x^6 - 3y^6$ and $4x^3 + 4y^3$.

Ans. $x^3 + y^3$.

9. Find the GCD of $x^3 + x^2 - ax^2 + a^2x - ax + a^2$ and $x^3 - x^2 - ax^2 + a^2x + ax - a^2$.

Ans. $x^2 - ax + a^2$.

10. Find the GCD of $x^{16} - y^{16}$ and $x^3 + x^2 + xy^2 + y^2$.

Ans. $x^2 + y^2$.

11. Reduce $\frac{x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4}{x^3 + 4ax^2 + 5a^2x + 2a^3}$ to its lowest terms.

Ans. GCD $x^2 + 2ax + a^2$.

Reduced fraction, $\frac{x^2 + 2ax + a^2}{x + 2a}$.

12. Reduce $\frac{x^3 + x^2 + x^2y^{-1} + xy^{-1} + 1 + x^{-1}}{x^2 - x + xy^{-1} - y^{-1} + x^{-1} - x^{-2}}$ to its lowest terms.

Ans. GCD $x^{-2} + y^{-1} + x$.

Reduced fraction, $\frac{x^2 + x}{x - 1}$.

13. Find the GCD of $x^3 + x^3y + 1 + y^2 + y + yx^{-3}$ and $x^3 + yx^3 + x^3y^2 + y^3 + y^2 + y$.

Ans. $x^3 + y$.

14. Reduce $\frac{x^2y + 2x^2 + 3x + xy + 1}{xy + 2x + 1 + yx^{-1} + y^2 + 2y}$ to its lowest terms.

Ans. GCD $x^{-1} + y + 2$.

Reduced fraction, $\frac{x^2 + x}{x + y}$.

Remainder in this example, $2x - y + 1 - xy - y^2x = 2x + 1 + xy - 2xy - y - y^2x = x(x^{-1} + y + 2) - yx(x^{-1} + 2 + y) = (x - yx)(x^{-1} + 2 + y)$.

Suppress factor, $x - yx$.

15. Reduce $\frac{1 + xy^{-2} + x^{-1} + x^{-2}y^2}{1 + xy^{-2} + x^{-2} + x^{-3}y^2}$ to its lowest terms.

Ans. GCD $x + y^2$.

Reduced fraction, $\frac{x^{-2} + y^{-2}}{x^{-3} + y^{-2}}$.

Remarks.

128. The last two examples show the importance of suppressing the *factor common to the remainder and not common* to the divisor.

Greatest common divisor of three or more polynomials.

129. The foregoing principles can be readily extended to finding the greatest common divisor of three or more polynomials. It is evident, that if we use one of the polynomials as a divisor, and divide all the others by it, that the remainders, if any, must contain the common divisor of all the polynomials. Suppress, in each of these remainders, the factors not common to the polynomial used as a divisor, and take that remainder which contains the lowest power of the arranged letter as the new divisor, and divide the other remainders by it; and continue in this way until we get an exact divisor; it will be the GCD sought. To illustrate by an example, find the GCD of

$$x^2 + ax - 2x - 2a,\ x^2 - x - 2, \text{ and } x^3 - 2x^2 - 4x + 8.$$

Use $x^2 - x - 2$ as the divisor, the remainder will be

$ax - x - 2a + 2$, or $a(x - 2) - (x - 2) = (a - 1)(x - 2)$, 0 and $-x^2 - 2x + 8$

Suppress the factor, $a - 1$, in the first remainder, and the result, $x - 2$, will exactly divide the other two remainders; hence, $x - 2$ is the greatest common divisor of the given polynomials.

Again, take

$x^2 + ax + x + a$, $x^3 + 3x^2 + 3x + 1$, and $x^3 + 2x^2 - ax^2 - 2ax + x - a$.

Use the first as the divisor; the three remainders will be

$0, -2a(x+1)+x+1+a^2(x+1)$, and $2a^2(x+1)-2a(x+1)$,

or 0, and $(x+1)(1+a^2-2a)$, and $(x+1)(2a^2-2a)$.

Hence, $x+1=$ GCD.

3d. Find the GCD of a^2-x^2, $a^2+2ax+x^2$ and $a^3+3a^2x+3ax^2+x^3$.

Ans. $a+x$.

4th. Find the GCD of x^6-y^6, x^9-y^9, $x^{15}-y^{15}$, and $x^{21}-y^{21}$.

Ans. x^3-y^3.

5th. Find the GCD of x^2-1, x^4-1, x^2-2x+1, and x^3-3x^2+3x-1.

Ans. $x-1$.

130. It will be seen that the GCD of any number of polynomials, as well as for two, may have its sign changed. Thus, the last GCD may be $x-1$, or $1-x$, and so for the others.

LEAST COMMON MULTIPLE.

131. The least common multiple of two or more quantities, is the least quantity that they will exactly divide. Thus, the least common multiple of 2, 4, and 6, is 12; of a^2, a, and b, is a^2b; of a^2+ab, b and b^2, is $a^2b^2+ab^3$, &c. It is plain that the product of all the quantities will be a common multiple of these quantities; that is, it will be exactly divisible by them. Thus, $2\times 4\times 6=24$ is a common multiple of 2, 4, and 6, but it is not their *least* common multiple. In like manner, a^3b is a common multiple of a^2, a, and b, but not their least common multiple. Multiply a common multiple by anything whatever, the product will still be a common multiple. Hence, there may be an infinite number of common multiples, but there can be but one least common multiple.

132. No quantity will be divisible by another, unless it contains all the factors of the second quantity. So, no quantity will be divisible by two or more quantities, unless it is divisible by each, and by all the factors of these quantities. To be the *least* common multiple of the given quantities, it must contain no more factors than they contain; and these factors must not enter to higher powers than in the given quantities. Thus, abc is not the least common multiple of a and b, because

it contains a factor, c, that they do not contain. Neither is a^2b the least common multiple, because the factor a enters to a higher power than in the given quantities. The expressions abc, and a^2b, are multiples, but not *least* common multiples.

133. If the given quantities contain a common factor, this must enter into the least common multiple raised to the highest power to which it is raised in the expressions to be divided, but it must not be repeated.

134. It must enter to the highest power, else the multiple which we formed would not be divisible by the expression containing the highest power of the common factor, and it must must not be repeated, else the multiple would not be the least common multiple.

135. From these principles we derive the following

RULE.

Decompose the given quantities into their prime factors, form a product composed of all the factors not common, and of the highest powers of the common factors, taking care to let no factor enter more than once. The product so formed will be the least common multiple required.

Form the least common multiple of ax^2, a^2x, bx^4, and a^3.

Decomposing, we have $a.x^2$, $a^2.x$, $b.x^4$, and a^3.

Hence, least common multiple, $a^3.x^4.b = a^3bx^4$.

It is plain that the common factor, a, must enter to the highest power (the third) to which it enters in one of the given quantities, else the multiple would not be divisible by that quantity. It is also plain that x must enter to the highest power, and that the factor b, not common, must also form part of the least common multiple, otherwise, the expression containing b would not be a divisor.

Form the least common multiple of 2, 8, 3, 9, a, a^2, and $5b$.

Decomposing into factors we have 2, 2^3, 3, 3^2, a, a^2, $5.b$.

Hence, $2^3.3^2.a^2.b.5 = 360a^2b$ is the least common multiple required.

By inspecting the result, 2^3, 3^2, $5a^2.b$, it is plain that it is the least common multiple. It is divisible by 2, because it contains a factor 2^3;

by 8, because it contains 2^3; by 3, because it contains 3^2, &c. Moreover, it is the least product that will divide the given quantities, for it is made up of the least factors that will fulfil the required conditions. The number 2 need not be raised to the third power to divide 2, but it must be to divide 8; so 3 need not be raised to the second power to divide 3, but it must be to divide 9.

We see, too, that no factor has been taken more than once.

3d. Find the least common multiple of $(a^2 + ax^2)$, $9a^3$, 21, and $7ab$.
Ans. $3^2 7\ (a + x^2)\ a^3 b = 63\ (a + x^2)\ a^3 b$.

4th. Find the least common multiple of $x^2 - y^2$, $x + y$, $7x - 7y$, and $x^3 - y^2 x$.
Ans. $7x\ (x^2 - y^2)$.

5th. Find the least common multiple of $3a^2b^3$, $4\ (x + 1)^2$, $16a^3b^2$, and $40\ (x + 1)b$.
Ans. $3.2^4.5\ (x + 1)^2 a^3 b^3 = 240\ (x + 1)^2 a^3 b^3$.

6th. Find the least common multiple of $2a$, $3a^2$, $4b$, $5b^2$, $6c$, $7c^3$, $8d^4$, $9d$, and $10abcd$.
Ans. $2^3.3^2.5.7a^2b^2c^3d^4 = 2520a^2b^2c^3d^4$.

Corollary.

136. The least common multiple of two or more quantities can also be found by dividing their product by the greatest common divisor. For, in the division of the product of the quantities by their greatest common divisor, the lowest powers of their common factors alone are divided out, and the quotient is made up of the factors not common, multiplied by those that are common, raised to the highest powers to which they enter in any of the given quantities, which, as we have seen, is the composition of the least common multiple.

137. Conversely, if we know the least common multiple of two or more quantities, we can find their greatest common divisor by multiplying the quantities together, and dividing their product by the least common multiple. For, let A, B, and C denote the quantities, D, their greatest common divisor, and L their least common multiple. Then, since $\frac{ABC}{D} = L$, $\frac{ABC}{L} = D$. This relation between the greatest common divisor and the least common multiple, enables us to verify our results in finding either.

LEAST COMMON MULTIPLE OF FRACTIONS.

138. It is often important to find the least common multiple of fractions, and as the rule for finding it for entire quantities fails in this case, it becomes necessary to demonstrate another rule.

Let us take the fractions

$$\frac{a^3}{bc^2}, \quad \frac{a^2}{b^2c^2}, \text{ and } \frac{a^2b}{c^2}.$$

We are required to find the least quantity that they will exactly divide, and give entire quotients. Now, to divide by a fraction, is to divide by the numerator, and multiply by the denominator. The quantity, then, that will be divisible by one of the fractions, as $\frac{a^3}{bc^2}$, must be divisible by a^3, or it will not be divisible after it has been multiplied by bc^2. And, as the same reasoning may be extended to the other fractions, the quantity sought must evidently be divisible by each of the numerators; and, in order to be the least quantity that will fulfil this condition, it must be the least common multiple of the numerators. But, since the denominators are to be multiplied by this least common multiple of the numerators, it is plain that a^3b cannot be the quantity sought. For, a^3b, the least common multiple of the numerators, divided by any quantity that will exactly divide the denominators, will be a smaller quantity than a^3b itself. For instance, $\frac{a^3b}{c}$ will be divisible by the given quantities, and be a smaller quantity than a^3b. It is plain, too, that a^3b, divided by the greatest quantity that will divide the denominators, will still be smaller than $\frac{a^3b}{c}$, and will be the smallest quantity that will be exactly divisible by the given quantities. Hence, $\frac{a^3b}{c^2}$ is the least common multiple of the fractions

$$\frac{a^3}{bc^2}, \quad \frac{a^2}{b^2c^2} \text{ and } \frac{a^2b}{c^2}.$$

RULE.

139. *Find the least common multiple of the numerators, and divide it by the greatest common divisor of the denominators; the fraction so formed will be the least common multiple of the given fractions.*

2. Find the least common multiple of $\frac{8}{15}$, $\frac{4}{45}$, and $\frac{12}{5}$.

Ans. $\frac{24}{5}$.

3. Find the least common multiple of

$$\frac{8\,(a + x^2)}{3}, \frac{2\,(a^2 + ax^2)}{9}, \text{ and } \frac{4a}{27}.$$

Ans. $\frac{8\,(a^2 + ax^2)}{3}$.

4. Find the least common multiple of

$$2a,\ 3a^2,\ \frac{4\,(a^4 - a^2x^2)}{3}, \frac{2\,(a - x)}{9}, \text{ and } \frac{6\,(a + x)}{5}.$$

Ans. $12a^2\,(a^2 - x^2)$.

5. Find the least common multiple of

$$\frac{7\,(x + y)^2}{3}, \frac{49\,(x + y)}{9}, \frac{21x^2}{27}, \text{ and } \frac{x + y}{12}.$$

Ans. $\frac{3\,(49)\,(x + y)^2x^2}{3} = 49\,(x + y)^2x^2$.

6. Find the least common multiple of

$$\frac{73a^2}{3}, \frac{73ab}{2}, \frac{x + y}{5}, \text{ and } \frac{a}{3}.$$

Ans. $73a^2b\,(x + y)$.

The demonstration being founded upon the hypothesis, that the fractions are reduced to their lowest terms, the rule is, of course, only applicable to such fractions.

GREATEST COMMON DIVISOR OF FRACTIONS.

140. The greatest common divisor of two or more fractions is the greatest quantity that will exactly divide them, giving entire quotients.

This quantity must be a fraction; for nothing but a fraction will divide a fraction reduced to its lowest terms, and give an entire quotient. The greatest common divisor of several fractions will then be a fraction itself; and since, when we divide a fraction by another fraction, we divide the numerator of the fraction assumed as the dividend by the numerator of the fraction taken as the divisor, and divide the denominator of the divisor by the denominator of the dividend, it follows, that the divisor sought must have a numerator that will divide each of the numerators of the given fractions, and a denominator that will be divisible by each of the given denominators. But, since the value of a fraction increases with the increase of its numerator and the decrease of its denominator, it is plain that a fraction, whose numerator would divide all the given numerators, and whose denominator would be divisible by all the given denominators, would not be the greatest fraction that will divide the given fractions, unless it has the greatest numerator and the least denominator that will fulfil the required conditions. Or, in other words, unless its numerator is the greatest common divisor of the given numerators, and its denominator the least common multiple of the given denominators.

RULE.

141. *Find the greatest common divisor of the numerators, and divide it by the least common multiple of the denominators; the fraction so formed will be the least common multiple of the given fractions.*

EXAMPLES.

1. Find the GCD of $\frac{2a^3}{3}, \frac{8a^3}{9}, \frac{4a^4}{27}$, and $6a^2$. *Ans.* $\frac{2a^2}{27}$.

2. Find the GCD of $\frac{7x^2}{12}, \frac{5x^3}{3}, \frac{(a+x)x}{4}$, and $\frac{x^4}{2}$. *Ans.* $\frac{x}{12}$.

3. Find the GCD of $\frac{2a}{3}, \frac{3}{4}, \frac{x^2}{12}$, and $\frac{x}{36}$. *Ans.* $\frac{1}{36}$.

4. Find the GCD of $\frac{3}{2a}, \frac{4}{3}, \frac{12}{x}$, and $\frac{36}{x}$. *Ans.* $\frac{1}{6ax}$.

5. Find the GCD of $\frac{12x}{7}, \frac{x^2}{21}, \frac{a^2x+x^2}{9}$, and $\frac{x^3}{3}$. *Ans.* $\frac{x}{63}$.

6. Find the GCD of $\frac{7}{2x}, \frac{21}{x^2}, \frac{9}{a^2x+x^2}$, and $\frac{3}{x^3}$.

Ans. $\frac{1}{12x^3(a^2+x)}$.

Remarks.

142. The greatest common divisor of entire quantities may be unity, and the quantities are then said to be prime with respect to each other. But it is evident that fractions can never be prime with respect to each

other; for, though their numerators may be prime, as in the 3d, 4th, and 6th examples, the least common multiple of their denominators can always be formed.

143. We may also remark that, as the least common multiple of entire quantities can always be formed; so, likewise, all fractions whatever have a least common multiple.

144. Knowing the least common multiple of any number of fractions, we can find their greatest common divisor by multiplying the fractions together, and dividing their product by their least common multiple.

145. Conversely, knowing their greatest common divisor, we can find their least common multiple by multiplying the fractions together, and dividing the product by the greatest common divisor.

EQUATIONS OF THE FIRST DEGREE.

146. An Equation is an expression containing two equal quantities, with the sign of equality between them. Thus, $x = a - b$ is an equation, and expresses, that the quantity represented by x is equal to the difference of the quantities represented by a and b.

147. The part on the left of the sign of equality is called the *first member* of the equation, and that on the right the *second member*.

148. Problems can be *stated* or expressed, and their solutions obtained by means of equations; that is, we can express, in algebraic language, the relation between an unknown quantity to be found, and one or more known quantities, and, by certain operations, can find the value of the unknown quantity.

149. The expressing the relation is called the *statement* of the problem; and the operation performed after the statement, to find the value of the unknown quantity, is called the *solution* of the problem. Thus, let it be required to find a quantity, which, being added once to itself, will give a sum equal to b. Let x be the unknown quantity; then, by the conditions, $x + x = b$, or, $2x = b$. Then, if twice x is equal to b, x itself must be equal to half of b, and $x = \frac{b}{2}$ is the final result. In this, making the equation, $x + x = b$ is the statement,

and the subsequent operation is the solution. If the value found for the unknown quantitity $\left(\frac{b}{2}\right)$ be substituted in the equation $x + x = b$, we will have $\frac{b}{2} + \frac{b}{2} = b$, a true equation. When the value found for the unknown quantity substituted in the equation of the problem makes the two members equal to each other, the equation is said to be *satisfied*, and we conclude that the solution is true.

150. The unknown quantity, the thing to be found, is usually represented by one of the final letters of the alphabet, x, y, and z. Known quantities are generally represented by the first letters of the alphabet, a, b, c, and d.

151. An equation with one unknown quantity is of the first degree, when the highest exponent of the unknown quantity in any term is unity; of the second, when the highest exponent of the unknown quantity in any term is two, &c.

$x + a = 5$ is an equation of the first degree.

$x^2 + x = a$ is an equation of the second degree.

$x^3 + x^2 + x = a$ is an equation of the third degree.

$x^4 + px^3 + mx^2 + nx = a$ is an equation of the fourth degree.

152. Equations which contain the unknown quantity from the highest to the first power inclusive, are called complete equations. The above are all complete equations with one unknown quantity.

153. Equations in which some of the powers of the unknown quantity are missing, are called incomplete equations. $x^2 = a$ is an incomplete equation of the second degree. $x^3 + x^2 = a$ is an incomplete equation of the third degree.

154. A *numerical* equation is one in which all the known quantities are represented by numbers. Thus, $x^2 + 2x = 4$ is a numerical equation.

155. A literal equation is one in which all the quantities, known and unknown, are represented by letters. Thus, $x^2 + ax = b$ is a literal equation.

156. An equation in which the known quantities are partly represented by letters, and partly by numbers, is a *mixed* equation. Thus, $x^2 + ax = b + 2$ is a mixed equation.

157. An identical equation is one in which the two members differ only in form, if they differ at all. Thus, $2x = \frac{4x}{2}$, $\frac{2x + a}{2} = x + \frac{a}{2}$, and $x = x$, are identical equations.

158. An *indeterminate* equation is one in which the value of the unknown quantity is indeterminate.

159. A single equation with two unknown quantities is necessarily indeterminate, for we must assume the value of one before we can determine that of the other. Thus, $x + y = 10$ is an indeterminate equation, because, by assuming $y = 1, 5, 4$, &c., we find $x = 9, 5, 6$, &c. And, by attributing an infinite number of arbitrary values to y, there will be an infinite number of values for x.

160. All identical equations are necessarily indeterminate. Thus, the equation $x = x$ will be true, or satisfied, when x is 1, 10, 1000, or anything whatever.

161. The *own* sign of a quantity is the sign with which the quantity is affected previous to any operation being performed upon it. The *essential* sign is the sign with which it is affected after the operation. Thus, multiply $+ a$ by -1; the own sign of a is plus, and its essential sign minus. But, multiply $+ a$ by $+ 1$, and the own and essential signs are the same. Subtract $+ b$ from a, then $a - (+ b) = a - b$. Here, b appears in the second member with the essential sign.

162. Since a positive quantity cannot be equal to a negative, and, conversely, it is evident that the essential sign of the two members of an equation must be the same.

163. Since quantities can only be equal to quantities of the same kind, it is plain that the two members of an equation must be composed of quantities of the same kind. Thus, if one member represent time, the other must represent time also.

These two principles are important, and ought to be remembered.

164. Several axioms, or self-evident propositions, are assumed as the basis of the principles by which equations are solved.

1. If equals be added to or subtracted from equals, the results will be equal.

2. If equals be multiplied or divided by equals, the results will be equal.

SOLUTION OF EQUATIONS OF THE FIRST DEGREE.

165. The transformation of an equation consists in changing its form, without destroying the equality of the two members.

There are three transformations of equations.

First Transformation.

166. The first transformation consists in clearing an equation of its fractions.

By the second axiom, we have a right to multiply both members of an equation by the same quantity; and it is evident that if we multiply the two members by the least common multiple of the denominators, the resulting equation will be free of fractions. For, since each denominator will divide the least common multiple, it is plain that the product of each numerator by the least common multiple will be divisible by each denominator. Thus, take the equation $\frac{x}{2} + \frac{x}{6} + 2 = 4$, multiply each member by 6, the least common multiple of the denominators, and the resulting equation is $3x + x + 12 = 24$.

It is plain, that if we multiply the two members by 12, the product of the denominators, the resulting equation will also be free from fractions. For each denominator will divide the product of 12 by its numerator, inasmuch as it will divide 12 itself. Multiplying by 12, we get $6x + 2x + 24 = 48$. Now, divide both members by 2, which we have a right to do by the second axiom, and we get $3x + x = 12$, the same equation as before.

Again, take the equation $\frac{x}{2} + \frac{x}{6} + \frac{x}{4} + 2 = 5$.

By the first method (the least common multiple being 12), we get $6x + 2x + 3x + 24 = 60$. By the second method, we get $24x + 8x + 12x + 96 = 240$. Divide both members by 4, and the results are the same. It will be seen that the second method is the same as multiplying each numerator into all the denominators except its own, and each entire quantity by the product of all the denominators.

RULE.

Multiply both members by the least common multiple, or by the product of the denominators.

When the least common multiple can be found without difficulty, the first method is preferable.

Second Transformation.

167. This consists in transposing known terms to the second member, and unknown terms to the first member.

Take the equation $2x - 2 = x + a$. (1)

By the first axiom, we have a right to add $+ 2$ to both members of the equation. Adding, we have $2x - 2 + 2 = x + a + 2$; or, since the $- 2$ and $+ 2$ destroy each other in the first member, the equation becomes $2x = x + a + 2$. By comparing this with the equation marked (1), we see that 2 has passed into the second member, by changing its sign.

Resume the equation $2x = x + a + 2$, and add minus x to both members. We get $2x - x = x - x + a + 2$; or, since $+ x$ and $- x$ destroy each other in the second member, $2x - x = a + 2$. Hence, x has passed into the first member by changing its sign.

We have demonstrated that quantities can be transposed from one member to another, provided we change their signs, since transposition is nothing more than adding to both members the quantities to be transposed, with their signs changed. The demonstration can be as readily made by subtracting the quantities with their appropriate signs. Hence, the second transformation is equivalent to adding to both members the quantities to be transposed, with their signs changed, or subtracting these quantities with their own signs from both members.

RULE.

Change the sign of each term transferred from one member to another.

Third Transformation.

168. The third transformation consists in clearing the unknown quantity of its coefficient or coefficients.

Resume the equation, $6x + 2x + 3x + 24 = 60$; then, by the second transformation, we have $6x + 2x + 3x = 60 - 24 = 36$. This indicates that the sum of $6x$, $2x$, and $3x$ is equal to 36; hence, obviously, $11x = 36$. Now, by the second axiom, we have a right to divide both members by the same quantity. Divide by 11, and we have $x = \frac{36}{11}$. The unknown quantity now stands unconnected with known terms, and we have its true value.

Take $$ax + bx - cx = m.$$

By the rules for factoring, we have $(a + b - c)\,x = m$; and by the second axiom, $x = \dfrac{m}{a + b - c}$.

These illustrations show that the third transformation is used when the equation has been cleared of its fractions, if it contained any, and when all the known quantities in the first member have been removed to the second, and when all the terms involving the unknown quantity have been placed in the first member, if not already there.

RULE.

Collect into a single algebraic sum all the coefficients of the unknown quantity, and divide the two members by this sum; if there is but one coefficient, divide both members by it.

EXAMPLES.

1. Clear the unknown quantity of its coefficient in the equation,

$$2x = b + c$$

Ans. $x = \dfrac{b + c}{2}$.

2. Clear the unknown quantity of its coefficient in the equation,

$$bx + 3x - cx = a.$$

Ans. $x = \dfrac{a}{b + 3 - c}$.

3. Clear the unknown quantity of its coefficient in the equation,

$$5x + ax - 2x = n.$$

Ans. $x = \dfrac{n}{3 + a}$.

4. Clear the unknown quantity of its coefficient in the equation,

$$7x + 3x + 2x = 12.$$

Ans. $x = 1$.

5. Clear the unknown quantity of its coefficient in the equation,

$$2x - 7x = a.$$

Ans. $x = \dfrac{a}{-5} = \dfrac{-a}{5}$.

169. It will be seen that the object of the three transformations is to free the unknown quantity of its connection with known terms, and make it stand alone in the first member of the equation. When so situated, its true value is known, being expressed by the second member of the equation.

170. The various steps in the solution of an equation, can be best illustrated by taking an example which will require all three transformations.

Take $$\frac{x}{2} + \frac{3x}{10} + a = b - 2x.$$

First step. To clear the equation of its fractions.

Result. $5x + 3x + 10a = 10b - 20x.$

Second step. To get the known terms to second member, and unknown to first member.

Result. $5x + 3x + 20x = 10b - 10a.$

Third step. To clear the unknown quantity of its coefficient.

Result. $x = \frac{10\,(b - a)}{28} = \frac{5\,(b - a)}{14}.$

171. Frequently, only two transformations are necessary in the solution of an equation; sometimes only one, but always at least one.

Take $$ax + b = c.$$

By second transformation, $ax = c - b$; and by third, $x = \frac{c - b}{a}$.

Take $2x - 3x = m$; then, by third transformation, $-x = m$.

In this the unknown quantity appears with the negative sign; but we were required to find the value of $+x$ and not of $-x$. Multiply both members by minus unity, which we have a right to do by the second axiom, and $-x$ becomes positive. Hence, $x = -m$ is the required value.

172. In general, when the sign of the unknown quantity is negative in the final result, and the conditions of the problem require it to be affected with the positive sign, both members must be multiplied by minus unity.

173. We observe, also, that the signs of all the terms of an equation

can be changed without altering the equality of the two members, since we have a right to multiply these members by minus unity.

174. From the foregoing principles we derive, for the solution of equations of the first degree, the following

RULE.

I. *If the equation contain fractions, clear it of them by multiplying both members by the product of the denominators, or their least common multiple.*

II. *Bring all the terms involving the unknown quantity to the first member (if not already there), and all the known terms to the second.*

III. *Collect into a single algebraic sum all the coefficients of the unknown quantity, and divide both members by this sum.*

IV. *If the unknown quantity appear in the final result with a negative sign, multiply both members by minus unity.*

EXAMPLES.

1. Solve the equation $x + 3x - \frac{1}{2}x = 14$. *Ans.* $x = 4$.

2. Solve the equation $4x + \frac{x}{2} - \frac{3x}{10} + \frac{4x}{5} = 50$.
Ans. $x = 10$.

3. Solve the equation $\frac{x}{5} + 3x + \frac{x}{15} + \frac{x}{3} - 60 = -6$.
Ans. $x = 15$.

4. Solve the equation $2x + 3x + 4x - 7 = -\frac{5}{2}$.
Ans. $x = \frac{1}{2}$.

5. Solve the equation $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} + \frac{x}{12} - \frac{24x}{7} = -\frac{176}{7}$.
Ans. $x = 12$.

6. Solve the equation $2x + 3x + 4x - 7 = -4$.
Ans. $x = \frac{1}{3}$.

7. Solve the equation $\frac{x}{a} + \frac{x}{c} + 3x = \frac{c}{a} + 3c + 1$.
Ans. $x = c$.

8. Solve the equation $nx + \frac{x}{n^2} + 4x - 5 = n^3 + 4(n^2 - 1)$.

Ans. $x = n^2$.

9. Solve the equation $\frac{x}{a+b} + 4x + b - a = 5b + 3a + 1$.

Ans. $x = a + b$.

10. Solve the equation $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} - \frac{x}{d} = abcd$.

$$\text{Ans. } x = \frac{(abcd)^2}{bcd + acd + abd - abc}.$$

11. Solve the equation $\frac{x}{2} - \frac{x}{4} + \frac{x}{p} + \frac{x}{m} - \frac{x}{a} = +$

$$\frac{(p^2 + pm + pa + 4p + 4m + 4a)}{4p} + \frac{p+a}{m} - \frac{(p+m)}{a}.$$

Ans. $x = p + m + a$.

12. Solve the equation $z + \frac{z}{a} + ab + az = m$.

$$\text{Ans. } z = \frac{a(m - ab)}{a^2 + a + 1}.$$

13. Solve the equation $y - \frac{y}{7} + \frac{y}{5} + a + by = a + \frac{2by}{3} - c$.

$$\text{Ans. } y = \frac{-105c}{111 + 35b}.$$

14. Solve the equation $\frac{7y}{3} + \frac{y}{a} - \frac{y}{6} + a + 3 - 6b = 10$.

$$\text{Ans. } y = 6a\,\frac{(7 + 6b - a)}{13a + 6}.$$

175. A few examples follow of equations in which the unknown quantity is affected with negative exponents, and in which the degree of the equation is apparently higher than the first.

1. Solve the equation $\frac{2x^{-1} + x - 1}{3} = \frac{2x + 4}{6x}$.

Ans. $x = 2$.

2. Solve the equation $\frac{3}{5x} + \frac{1}{5} = 2$.

Ans. $x = \frac{1}{3}$.

3. Solve the equation $\frac{x}{5} + \frac{x^{-2}}{5} = \frac{5x^{-3} + x}{5}$.

Ans. $x = 5$.

4. Solve the equation $\frac{x^3+7x^2-3x}{3} = \frac{12x^2+14x^2}{6} - x.$

Ans. $x = 6.$

5. Solve the equation $\frac{x^{-1}}{14} + \frac{x}{7} - \frac{2x^{-2}}{7} = \frac{1}{7x^{-1}}.$ *Ans.* $x = 4.$

6. Solve the equation $x^{-1} + \frac{1}{ax} + \frac{b}{x} = \frac{a+1+ab}{bx^2}.$

Ans. $x = \frac{a}{b}.$

7. Solve the equation $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{20} + \frac{1}{5} = 80x^{-1}.$

Ans. $x = 40.$

8. Solve the equation $ax^{-2} + \frac{x^{-1}}{a} = \frac{b}{x} + \frac{c}{x^2}.$

Ans. $x = \frac{a(c-a)}{1-ab}.$

9. Solve the equation $2x^{-3} + \frac{3}{x^2} + \frac{4}{x^2} = x^{-2} + \frac{8}{x^2}.$

Ans. $x = 1.$

10. Solve the equation $2x^{-2} = \frac{6ax^{-3}}{3} + \frac{2m}{x^3}.$

Ans. $x = a + m.$

176. *Solutions of problems producing equations of the first degree with an unknown quantity.*

It has been remarked, that the solution of a problem consists of two distinct parts, the statement and the solution.

The examples just given, illustrate the manner of solving the equation after the statement has been made.

No general rules can be given for making the statement, which depends mainly upon the ingenuity of the student.

Sometimes there is but a single thing to be determined, and in that case, by representing it by one of the final letters of the alphabet, and following the conditions of the problem, we get the equation or statement required.

Often, however, two or more things are to be determined, then the undetermined quantity, upon which the other quantities depend, is represented by one of the final letters of the alphabet.

As an example of the first class of problems, let it be required to find a quantity, which, added to half itself, will give a sum equal to 6.

Let x represent the quantity, then half of the quantity will be represented by $\frac{1}{2}x$, and by the conditions we have $x + \frac{1}{2}x = 6$. Solving the equation, we get $x = 4$.

As an example of the second class of problems, take the following: A man invests half of his money in State stocks, and a third of the remainder in a Savings' Institution, and has four hundred dollars left. Required the entire amount of his capital, and the amount of each of his two investments.

Since his investments depend upon his capital, let $x =$ capital. Then $\frac{1}{2}x =$ amount invested in State stock, and $\frac{x - \frac{1}{2}x}{3} = \frac{1}{6}x =$amount placed in Savings' Institution. The remainder, after these investments, is plainly $x - (\frac{1}{2}x + \frac{1}{6}x)$; but this remainder is, by the conditions of the problem, equal to four hundred dollars. Hence, $x - (\frac{1}{2}x + \frac{1}{6}x) = 400$. Solving, we get $x = 1200$.

Then,	$\frac{1}{2}x =$	600,	Investment in State stock.
	$\frac{1}{6}x =$	200,	" in Savings' Institution.
		400,	The amount uninvested.
		1200,	Original capital.

The solution has been verified by adding the two investments to what was left, and getting a sum equal to the original capital. The solutions of all problems ought to be verified in a similar manner.

We have seen that an equation is said to be satisfied when the value found for the unknown quantity substituted in the given equation, makes the two members equal to each other

But the value of the unknown quantity in the solution of a problem must not only satisfy the equation of the problem (the statement), but must also fulfil the required conditions. In consequence of the use of negative quantities in Algebra, the equation of the problem (the statement), is often satisfied, when the required conditions are not fulfilled in a strict, *arithmetical* sense.

As an illustration, let it be required to find a quantity, which, when added to 4, will give a sum equal to 2. Let x be the quantity, then $x + 4 = 2$. Hence, $x = -2$. Now, -2 will satisfy the equation of the problem, but will not fulfil the conditions in an *arithmetical* sense; for we were required to find a quantity, which, when *added* to 4, would give a sum equal to 2, and the quantity found is really to be taken from 4.

It will be seen hereafter, that there are several cases in which the equation may be satisfied, and the conditions not fulfilled.

EXAMPLES.

A farmer has two kinds of oats; the one kind worth 30 cents per bushel; the other 20 cents per bushel. He mixes 50 bushels of the superior article with a certain amount of the inferior, and sells the mixture at 22 cents per bushel. How much of the second quality did he put with the first in order to make the rate of sale the same?

Let x represent the amount of the inferior article; then, by the condition, $50 \cdot 30 + 20x = (50 + x)\,22$. Hence, $x = 200$. In this example, the verification of the equation and the fulfilment of the conditions are the same.

2. A farmer has 50 bushels of oats, worth 30 cents per bushel, and 200 bushels, worth 20 cents per bushel. He mixes the two lots together. At what price ought he to sell the mixture?

Let x represent the required price of the mixture per bushel; then, $50 \cdot 30 + 20 \cdot 200 = 250x$. Hence, $x = 22$.

3. What number must be subtracted from the numerator and denominator of the fraction $\frac{3}{4}$, to make the new fraction the reciprocal of the old? *Ans.* $x = 7$.

4. What number must be subtracted from the numerator and denominator of any fraction $\frac{m}{n}$ to make the new fraction, the reciprocal of the old. *Ans.* $x = m + n$.

The solution is general, and is true for any fraction whatever.

5. What number, added to the numerator and denominator of the fraction $\frac{3}{5}$, will make the new fraction 4 times as great as the old?

Ans. $x = -3\frac{3}{5}$.

What does this value satisfy?

6. What number, added to the numerator and denominator of any fraction $\frac{m}{n}$, will make the new fraction 4 times as great as the old?

Ans. $x = \dfrac{3mn}{n - 4m}$.

7. What number, added to any fraction $\frac{m}{n}$, will give a sum equal to the numerator. *Ans.* $x = \dfrac{m\,(n-1)}{n}$.

8. What number, added to any fraction $\frac{m}{n}$, will give a sum equal to the denominator?

$$Ans.\ x = \frac{n^2 - m}{n}.$$

In what case will this result fail to satisfy the conditions of the problem?

9. What number, multiplied by any fraction $\frac{m}{n}$, will give a product equal to the sum of the numerator and denominator?

$$Ans.\ x = \frac{(m + n)\,n}{m}.$$

How may the last four results be made applicable to any fraction whatever?

10. In a square floor of a college building there is a certain number of brick; if one more brick is added to each side of the floor, and the square form be preserved, there will be 61 more brick than at first. How many brick does the floor contain? *Ans.* $x^2 = 900$.

11. Two pipes lead into a reservoir capable of containing 160 hogsheads of water. The first pipe can fill it in 16 hours, and the two together in 12 hours. In what time can the second alone fill it?

Ans. $x = 48$ hours.

12. Two travellers travel at the same rate, the one starting before the other; at 12 o'clock the first has travelled 4 times as far as the second, but when they had each gone 15 miles further, the entire distance passed over by the first was only double that of the second. Required these distances. *Ans.* 45 and $22\frac{1}{2}$ miles.

12. The sum of two numbers is a, and their difference b. Required the two numbers.

Ans. The greater, $\frac{a}{2} + \frac{b}{2}$, the smaller $\frac{a}{2} - \frac{b}{2}$.

We see from this result that, knowing the sum and difference of two quantities, we get the greater by adding the half sum to the half difference, and the less, by subtracting the half difference from the half sum.

This formula is of extensive application, and ought to be remembered.

13. A Californian gold digger wishes to sell a vessel full of gold mixed with sand. The vessel, when filled with gold, will weigh ten pounds, and when filled with sand, one pound; the mixture weighs seven pounds. How much gold is in it? *Ans.* $6\frac{4}{11}$ lbs.

14. The railing around the altar of the Cathedral in the City of Mexico is a composition of gold and silver. Assuming that 279 pounds of the composition loses 20 pounds when immersed in water, and that $19\frac{1}{2}$ pounds of gold lose one pound in water, and $10\frac{1}{4}$ pounds of silver lose one pound in water. Required the proportion of gold and silver in the alloy. *Ans.* Gold : Silver : : 156 : 123.

15. Two men purchase together a barrel of flour, weighing 196 pounds, for 5 dollars; the first pays half a dollar more than the second. How ought they to divide the flour?

Ans. The first ought to have $107\frac{4}{5}$ lbs, the second $88\frac{1}{5}$.

GEOMETRICAL PROPORTION.

177. Ratio is the quotient arising from dividing one quantity by another of the same kind. Thus, if M and N represent quantities of the same kind, then $\frac{M}{N}$ expresses the ratio of M to N.

Four quantities, M, N, P, and Q, are said to be proportional when the ratio of the first to the second is the same as that of the third to the fourth.

Thus, 64, 8, 16, and 2 constitute a proportion, because the first, divided by the second, is equal to the third divided by the fourth.

The proportionality of four quantities, M, N, P, and Q, is expressed thus, M : N : : P : Q, and is read, M is to N as P is to Q.

The first and last terms of a proportion are called the *extremes;* the second and third the *means.* Of four proportional quantities, the first and third are called *antecedents;* the other two, *consequents.* Of three quantities, M, N, and P; when M : N : : N : P, then N is said to be a *mean* proportional between M and P; and N is, at the same time, antecedent and consequent.

Two quantities, M and N, are said to be reciprocally proportional, when the one increases as fast as the other diminishes. One of the quantities must be equal to a fraction with a constant numerator, and with the other quantity for its denominator.

Thus, $M = \frac{4}{N}$ expresses that M and N are reciprocally proportional.

Equi-multiples of two quantities are the products which arise from multiplying them by the same quantity. Thus, aM and aN are equi-multiples of M and N, the common factor being a.

Theorem I.

If four quantities are in proportion, the product of the extremes will equal the product of the means.

For, when M : N : : P : Q, we know that $\frac{M}{N} = \frac{P}{Q}$; and by clearing of fractions, MQ = PN.

This theorem furnishes an important test of the proportionality of four quantities. Whenever the product of the extremes is equal to that of the means, the proportion is true; and when that is not the case, it is a false proportion.

Theorem II.

178. When the product of two quantities is equal to the product of two other quantities, two of them may be taken as the extremes, and two as the means of a proportion.

For, suppose MQ = NP, divide both members by QN, the first member will become $\frac{M}{N}$, and the second $\frac{P}{Q}$, and we have $\frac{M}{N} = \frac{P}{Q}$. From which we get the proportion M : N : : P : Q.

Let 2 . 4 = 8 . 1. Then 2 : 8 : : 1 : 4.

Theorem III.

179. The square of a mean proportional is equal to the product of the other two terms of the proportion.

For, from the definition of a mean proportional, we have M : N : : N : P; hence, by Theorem I., N^2 = MP.

Let 2 : 4 : : 4 : 8. Then $4^2 = 2 . 8 = 16$.

Theorem IV.

180. When four quantities are in proportion, they will also be in proportion by *alternation;* that is, when antecedent is compared with antecedent, and consequent with consequent.

For, let M : N : : P : Q; then, by Theorem I., MQ = NP, and dividing this equation, member by member, by PQ, we get $\frac{M}{P} = \frac{N}{Q}$, from which M : P : : N : Q.

Let 4 : 8 : : 12 : 24. Then, also, 4 : 12 : : 8 : 24.

Theorem V.

181. When four quantities are in proportion, they will be in proportion when taken *inversely;* that is, when the consequents take the place of antecedents, and the antecedents the place of consequents.

Let M : N : : P : Q; then, also, we have MQ = NP, or NP = MQ; divide both members by MP, and we get $\frac{N}{M} = \frac{Q}{P}$, from which N : M : : Q : P.

Theorem VI.

182. If four quantities are in proportion, they will be in proportion by *composition;* that is when the sum of antecedent and consequent is compared with either antecedent or consequent.

Let M : N : : P : Q; then, also, $\frac{M}{N} = \frac{P}{Q}$. Add 1 to both members, and reduce to a common denominator, and we have $\frac{M + N}{N} = \frac{P + Q}{Q}$, from which we get M + N : N : : P + Q : Q.

Let 4 : 12 : : 8 : 24. Then 4 + 12 : 12 : 8 + 24 : 24.

In like manner it may be shown, that when M : N : : P : Q, that we will also have M + N : M : : P + Q : P.

Theorem VII.

183. When four quantities are in proportion, they will also be in proportion by *division;* that is, when the difference of antecedent and consequent is compared with either antecedent or consequent.

Let M : N : : P : Q; then, also, we have $\frac{M}{N} = \frac{P}{Q}$. Subtract 1 from both members, and reduce to a common denominator, we will have $\frac{M - N}{N} = \frac{P - Q}{Q}$, from which we get M — N : N : : P — Q : Q.

Let 12 : 4 : : 24 : 8. Then 12 — 4 : 4 : : 24 — 8 : 8.

It may be shown in like manner, that when four quantities are in proportion, M : N : : P : Q, that we will also have M — N : M : : P — Q : P.

Let 12 : 4 : : 24 : 8. Then, also, 12 — 4 : 12 : : 24 — 8 : 24.

Theorem VIII.

184. Equi-multiples of any two quantities are proportional to the quantities themselves.

For, take the identical proportion, M : N : : M : N; then MN = NM. Multiply both members by m, and there results mM . N = mN . M; and, since mM and mN may be regarded as single terms, we have from Theorem 2d, M : N : : mM : mN.

Let 1 and 2 be multiplied by the same number, 3; then 1 . 2 : : 1 . 3 : 2 . 3; or 1 . 2 : : 3 : 6.

Theorem IX.

185. If equi-multiples of the antecedents of four proportional quantities, and also equi-multiples of the consequents be taken, the four resulting quantities will be proportional.

For, let M : N : : P : Q; then MQ = NP. Multiplying both members by mn, and there results mM . nQ = nN . mP. Hence, mM : nN : : mP : nQ.

Let 1 : 2 : : 4 : 8, and take $m = 4$, and $n = 3$.

Then 4 . 1 : 3 . 2 : : 4 . 4 : 3 . 8; or 4 : 6 : : 16 : 24.

It is plain that the above theorem is equally true when $m = n$. So, that all the terms of a proportion may be multiplied by the same quantity without destroying the proportionality of the terms. An infinite number of proportions may then be formed from a single proportion.

Theorem X.

186. If there be two sets of four proportional quantities, having two terms, the same in both, the remaining terms will constitute a proportion.

Let M : N : : P : Q.
M : N : : R : S.

Then, MQ = NP, or NP = MQ.
MS = NR and NR = MS.

Dividing these equations member by member, we get $\frac{P}{R} = \frac{Q}{S}$, from which, P : R : : Q : S, or P : Q : : R : S. (Art. 180).

And the same property can evidently be shown when any other two terms are equal.

Let 1 : 2 : : 4 : 8.
1 : 2 : : 6 : 12. Then 4 : 6 : : 8 : 12, or 4 : 8 : : 6 : 12.

Let 1 : 2 : : 4 : 8.
1 : 3 : : 4 : 12. Then 2 : 3 : : 8 : 12, or 2 : 8 : : 3 : 12.

It will be seen that the corresponding terms of the proportion must be taken in connection with each other.

Theorem XI.

187. Of four proportional quantities, if the two antecedents be augmented or diminished by quantities which are proportional to the two consequents, the resulting quantities will be proportional to the consequents.

For, let M : N : : P : Q.
N : Q : : R : S.

Then, $MQ = NP$. (A).
$NS = QR$.

Or, $QR = NS$. (B).

Adding and subtracting (B) from (A), there results $Q (M \pm R) = N (P \pm S)$. Hence, $M \pm R : P \pm S : N : Q$.

Let 1 : 2 : : 4 : 8.
And 3 : 6 : : 4 : 8. Then $1 \pm 3 : 2 \pm 6 : : 4 : 8$.

It can be shown in like manner, that if the two consequents be augmented or diminished by quantities, which are proportional to the antecedents, the resulting quantities will be proportional to the antecedents.

The reciprocal of a quantity is unity divided by the quantity; thus, the reciprocal of A is $\frac{1}{A}$.

Theorem XII.

188. Two quantities are inversely proportional to their reciprocals.

For, take the identical proportion A : B : : A : B; then $AB = BA$.

Divide both members by AB; then

$$A\not{B} \cdot \frac{1}{A\not{B}} = B\not{A} \cdot \frac{1}{\not{A}B}, \text{ or } A\frac{1}{A} = B \cdot \frac{1}{B}.$$

Hence, $A : B :: \frac{1}{B} : \frac{1}{A}$. Thus, $2 : 3 :: \frac{1}{3} : \frac{1}{2}$.

Theorem XIII.

189. If there are any number of proportions having the same ratio, any one antecedent will be to its consequent as the sum of all the antecedents to the sum of all the consequents.

For, let

$$\left.\begin{array}{l} M : N :: P : Q. \\ M : N :: A : B. \\ M : N :: C : D. \\ M : N :: E : F. \end{array}\right\} A$$

Then,

$$\begin{array}{l} MQ = NP. \\ MB = NA. \\ MD = NC. \\ MF = NE. \end{array}$$

Adding these equations member by member, we get $M(Q + B + D + F) = N(P + A + C + E)$.

Hence, $M : N :: P + A + C + E : Q + B + D + F$.

Let

$$\begin{array}{l} 1 : 2 :: 4 : 8. \\ 1 : 2 :: 6 : 12. \\ 1 : 2 :: 8 : 16. \end{array}$$

Then, $1 : 2 :: 4 + 6 + 8 : 8 + 12 + 16$, or $1 . 2 :: 18 : 36$.

It is not necessary that the first antecedent and consequent of each proportion should be M and N; all that is required is, that they have the same ratio; for, then, we can substitute M and N for them, and the above demonstration becomes applicable. To show our authority for this substitution, take the two proportions,

$$M : N :: P : Q, \text{ and } R : S : T : U.$$

From the second, we get $RU = ST$, or $\frac{R}{S} = \frac{T}{U}$; and, since, by hypothesis, $\frac{R}{S} = \frac{M}{N}$, there results $\frac{M}{N} = \frac{T}{U}$. Hence, $M : N :: T : U$.

When we have any number of proportions, like those marked A, having a common ratio, we can obviously form a continued proportion from them.

Thus, $M : N :: P : Q :: A : B :: C : D :: E : F.$

So, also, $1 : 2 :: 4 : 8 :: 6 : 12 :: 8 : 16.$

The character : : is written before each new antecedent, and refers it and its consequent back to the first antecedent and consequent.

It is obvious that any antecedent may be repeated any number of times, provided, that its consequent is repeated the same number of times; and, also, that any consequent may be multiplied or divided by anything whatever, provided, that the same operation be performed on its consequent.

Thus, $M : N :: P + A + C + E : Q + B + D + F,$
may be written,

$$M : N :: P + A + A + C + mE : Q + B + B + D + mF,$$

without altering the truth of the proposition.

Thus, let $4 : 8 : 6 : 12$, and $4 : 8 :: 5 : 10$.

Then, $4 : 8 :: 6 + 5 : 12 + 10$, and $4 : 8 :: 6 + 5 + 5 : 12 + 10 + 10$.

And also, $4 : 8 :: 6 \times 9 + 5 + 5 : 12 \times 9 + 10 + 10$.

Theorem XIV.

190. If four quantities are in proportion, the sum of the first antecedent and consequent will be to their difference as the sum of the second antecedent and consequent is to their difference.

For, since $M : N :: P : Q$, it follows that $MQ = NP$.

Add NQ to both members, then $Q (M + N) = N (P + Q)$.

Subtract NQ from both members, then $N (P - Q) = (M - N) Q$.

Multiplying the two equations together, there results

$$(M + N) (P - Q) = (P + Q) (M - N).$$

From which we get $M + N : M - N :: P + Q : P - Q$.

Let $2 : 6 :: 8 : 24$.

Then $2 + 6 : 2 - 6 :: 8 + 24 : 8 - 24$, or $8 : -4 :: 32 : -16$.

Theorem XV.

191. If there are any number of proportional quantities, a single proportion may be formed from them by multiplying together the corresponding terms of all the proportions.

For, let
$$M : N : : P : Q.$$
$$A : B : : C : D.$$
$$E : F : : G : H.$$

Then,
$$MQ = NP.$$
$$AD = BC.$$
$$EH = FG.$$

Multiplying these equations, member by member, we get

$$(MAE)\ (QDH) = (NBF)\ (PCG).$$

Hence,
$$MAE : NBF : : PCG : QDH.$$

Let
$$1 : 2 : : 4 : 8.$$
$$3 : 1 : : 9 : 3.$$
$$5 : 10 : : 2 : 4.$$

Then, also,
$$15 : 20 : : 72 : 96.$$

Theorem XVI.

192. If four quantities are in proportion, their like powers will be proportional.

For, let
$$M : N : : P : Q.$$

Then,
$$MQ = NP.$$

Squaring both members, $M^2Q^2 = N^2P^2$.

Hence,
$$M^2 : N^2 : : P^2 : Q^2.$$

In like manner it may be shown that $M^m : N^m : : P^m : Q^m$.

Let
$$1 : 2 : : 4 : 8.$$

Then, also, $1^2 : 2^2 : : 4^2 : 8^2$, or $1 : 4 : : 16 : 64$.

Likewise, $1^3 : 2^3 : : 4^3 : 8^3$, or $1 : 8 : : 64 : 512$.

And similar proportions can be obtained for the 4th, 5th, &c., powers.

General Remarks.

193. All the foregoing theorems have been deduced from the first two; and it is obvious that an infinite series of proportions might be deduced from them. The test of the truth of any proportion deduced directly or indirectly from the first two theorems, is the product of the extremes being equal to the product of the means. Thus, if

$$M : N :: P : Q; \text{ then, also, } M + 4 : P + 4\frac{Q}{N} : N : Q.$$

For, by multiplying the extremes and means together, we have

$$MQ + 4Q = NP + 4Q,$$

which is a true equation when $MQ = NP$. In general, no proportion is false, however absurd it may seem, when the product of the extremes is equal to the product of the means. Thus, $1 : x^4 :: x^{-4} : 1$ is a true proportion, because the product of the extremes is equal to unity, and that of the means also equal to unity.

The proportion $M : N :: P : Q$, gives $MQ = NP$, from which we can get the four equations

$$\frac{M}{N} = \frac{P}{Q},$$

$$\frac{M}{P} = \frac{N}{Q},$$

$$\frac{Q}{N} = \frac{P}{M},$$

$$\frac{Q}{P} = \frac{N}{M},$$

involving eight distinct ratios. And, since ratio is the quotient arising from dividing one quantity by another of the same kind, we conclude, from the above series of equations, that either the first consequent or the second antecedent must represent quantities of the same kind as the first antecedent; and that either the first antecedent or the second consequent must represent quantities of the same kind as the first consequent. We also see that two distinct species of quantities, and but two, can be represented by a proportion. If two terms of a proportion are abstract numbers, the proportion can represent but one kind of quantity.

194. In the equation $MQ = NP$, resulting from the proportion $M : N :: P : Q$, if three terms are known, it is plain that the fourth term can be found by solving the equation with reference to it. In the Single Rule of Three of Arithmetic, three terms are given to find the fourth. Thus, suppose the first three terms of a proportion are 3, 4, and 6. The fourth term can be found from the proportion $3 : 4 :: 6 : x$. Hence $3x = 24$, or $x = 8$, and the complete proportion is $3 : 4 :: 6 : 8$. So, likewise, let the first three terms be a, b, and c, then the fourth results from the proportion, $a : b :: c : x$. Hence, $x = \frac{bc}{a}$. And we see that the fourth term can be found by multiplying the second and third together, and dividing their product by the first.

If the first, or either of the middle terms is unknown, we have only to represent the unknown term by x, form the equation from the proportion, and then solve it with reference to x.

Let the last three terms of a proportion be a, b, and c, to find the first. We have, then, $x : a :: b : c$. Hence, $x = \frac{ab}{c}$.

So, we see that either extreme can be found by multiplying together the means, and dividing their product by the other extreme.

Let the second term be unknown, and the other three be a, b, and c. Then $a : x :: b : c$. Hence, $x = \frac{ac}{b}$. Let the third term be unknown, and the other three be a, b, and c. Then $a : b :: x : c$. Hence, $x = \frac{ac}{b}$. We see that either mean may be found by multiplying the extremes together, and dividing their product by the other mean.

EXAMPLES.

1. The first term of a proportion is a^2; the third, c^2; the fourth, m^2. What is the second term?

Ans. $\frac{a^2m^2}{c^2}$.

2. The first term of a proportion is a^{m+n}; the second, a^{-n}; the fourth, a^{-m}. What is the third term? *Ans.* a^{2n}.

3. The three last terms of a proportion are a^r, b^s, and $(ab)^{rs}$. What is the first term? *Ans.* $a^{-s}b^{-r}$.

4. Given the first three terms of a proportion, $a^2 - b^2$, $a + b$, and $a - b$, to find the fourth term. *Ans.* 1.

5. Given three first terms of a proportion, a^r, b^r, and c^r, to find the fourth. *Ans.* $b^r c^r a^{-r}$.

PROBLEMS IN GEOMETRICAL PROPORTION.

195. 1. Two numbers are to each other as 2 to 3; but, if 8 be added to both, the sums thence arising will be to each other as 4 to 5. What are the numbers? *Ans.* 8 and 12.

Let $x =$ smaller, then the greater will be $\frac{3}{2}x$. For, $2 : 3 :: x$ to fourth term $\frac{3}{2}x$. Then $x + 8 : \frac{3}{2}x + 8 :: 4 : 5$.

2. An author can write 240 pages in ten weeks: how many can he write in $12\frac{1}{2}$ weeks? *Ans.* 300 pages.

3. Two quantities are to each other as a is to b; but, if c be added to both of them, the resulting sums will be to each other as d to e. What are the quantities?

$$Ans.\ \frac{(e - d)\,ac}{bd - ae} \text{ and } \frac{(e - d)\,bc}{bd - ae}.$$

4. An author can write a pages in b weeks: suppose that he writes uniformly, how many pages can he write in c weeks?

$$Ans.\ x = \frac{ac}{b} \text{ pages.}$$

What values must be given to the letters in order to make Examples 3 and 4, the same as 1 and 2? How can the results in the last two examples be verified?

5. A father's age is now 3 times that of his son, but in 10 years more the father's age will only be double that of the son. What are their respective ages now? *Ans.* Father, 30; son, 10.

6. A father's age is now a times greater than that of his son; but in b more years the father will only be c times older than his son. What are the respective ages of father and son?

$$Ans.\ \text{Father, } \frac{ab\,(c - 1)}{a - c};\ \text{son, } \frac{(c - 1)\,b}{a - c}.$$

7. A gentleman puts out his money at a certain rate of interest, and derives \$500 income. He puts the same sum out a second year, one per cent. more advantageously, and derives \$600 from it. What are the two rates of interest? *Ans.* 5 and 6 per cent.

8. The governor of a besieged town has provisions enough to allow each of the garrison $1\frac{1}{2}$ pounds of bread per day, for 60 days; but, as he learns that succor cannot be expected for 80 days, he finds it necessary to diminish the allowance. What must the allowance be?

Ans. $1\frac{1}{8}$ lbs. per day.

9. A gentleman, who owns 20 slaves, had laid in a twelve-months' supply for them, when he purchased 10 more. How long will his supplies last? *Ans.* 8 months.

10. A planter, who knows that his negro-man can do a piece of work in 5 days, when the days are 12 hours long, asks how long it will take him when the days are 15 hours long. *Ans.* 4 days.

11. Three negroes can hoe a field of cotton in 7 days: how long will it take 4 to do the same work? *Ans.* $5\frac{1}{4}$ days.

12. A family of 5 persons use a barrel of flour in 6 weeks: how long will 2 barrels last 7 persons, using it at the same rate.

Ans. $8\frac{4}{7}$ weeks.

13. Two farmers purchase a piece of land for \$1000; whereof the first pays \$600, and the second \$400, and sell it again for \$1200. What proportion of the profit ought each to receive?

Ans. The first, \$120; the second, \$80.

14. A bookseller purchased a certain number of books at the rate of 2 for a dollar, and also an equal number at the rate of 3 for a dollar, and sold the whole at the rate of 5 for 3 dollars, and gained \$22 by the sale. What was the entire number of books he bought and sold?

Ans. 120.

15. The hour and minute hands of a clock are together at 12 o'clock. When will they be together again?

Ans. $5\frac{5}{11}$ minutes past one o'clock.

Let x be the space after one passed over by the hour hand before being overtaken by the minute hand. Then, $12 : 1 :: 60 + x : x$.

16. Two pedestrians start from the same point at the same time, to walk around a race-course a mile in circumference. The first walks 11 yards in a minute, and the second 34 yards in 3 minutes. How many times will the first have gone around the track before he is overtaken by the second? *Ans.* 33 times.

17. A brick-mason has brick 9 inches long, and $4\frac{1}{2}$ inches wide, with which to build a wall $112\frac{1}{2}$ feet long. He wishes to place 180 bricks in a row, and to lay some of them with their ends to the front

as *headers*, and some of them lengthwise as *stretchers*. How many headers and how many stretchers must there be in each row?

Ans. 60 headers, and 120 stretchers.

18. A brick-mason wishes to place twice as many stretchers as headers in a wall 105 feet long. How many of each kind must he use, supposing the dimensions of the brick the same as in the last problem?

Ans. 112 stretchers, and 56 headers.

19. A Freshman recited 5 times a week in mathematics, and his average for the week was 66. His average for the first three days was to the average for the last two as 7 to 6. What were those averages?

Ans. 70 and 60.

20. Three farmers purchase 900 acres of land for $9000; of which the first pays $2000, the second $3000, and the third $4000. What share ought each to get, supposing that the land is equally valuable throughout?

Ans. The first, 200 acres; the second, 300, and the third, 400.

21. A composition of copper and tin, containing 50 cubic inches, was found to weigh 220·7 ounces. Assuming that a cubic inch of copper weighs 4·66 ounces, and that a cubic inch of tin weighs 3·84 ounces, what must have been the relative proportion of tin and copper in the composition? *Ans.* Tin to the copper as 1 to $2\frac{1}{3}$.

NEGATIVE QUANTITIES.

196. Quantities are considered as negative when opposed, in character or direction, to other quantities of the same kind that are assumed to be positive.

If a ship, sailing at the rate of 10 miles per hour, encounter a head-wind that drives it back at the rate of 8 miles per hour, then its rate of advance will plainly be expressed by 10 — 8 miles per hour. Here the retrograde movement, as opposed to the forward, is considered negative. Suppose the wind to increase in violence until the ship is carried back at the rate of 12 miles per hour. Then its rate of advance will be expressed by 10 — 12, or — 2 miles per hour. The ship will then plainly be carried back at the rate of 2 miles per hour, and we see that the minus sign has indicated a change of direction.

If a man be now thirty years old. His age, four years hence, will be expressed by $30 + 4$. Four years ago, it would have been expressed by $30 - 4$. Here, future time being positive, past time is negative. And we see that a change of character is again indicated by a change of sign.

If a man's money and estate be represented by a, and his debts by b; then $a - b$ will express what he is worth. His debts, as opposed to his property, are considered negative. If b be greater than a, then $a - b$ is negative; and we commonly say that the man is worth less than nothing. We see that there has been a change of sign in the expression for what the man was worth, corresponding to a change in the estimation of that worth. When the worth fell below zero, its expression became negative, because regarded as positive when above zero.

If a man agree to labor for two dollars a day, and to forfeit one dollar for every day that he is idle; and he labor 4 days, and is idle 2; then his wages will be $4 \times 2 + 2\,(-1)$, or $4 \times 2 - 2\,(+1) = 8 - 2$ dollars. We see that we have regarded as negative either the forfeiture, as opposed to the gain, or the working days as opposed to the idle.

Distance, regarded as positive, when estimated in one direction, must be considered negative when estimated in a contrary direction.

Let a distance, AB, estimated towards the right from the point A, be represented by $+ m$; and let a distance, BC, estimated towards the right from the point B, be represented by $+ n$.

Then, $$AC = AB + BC = m + n.$$

Now, suppose the point C be made to fall between A and B.

Then, $$AC = AB - BC = m - n.$$

And we see that the expression for BC has become negative, and that this change of sign corresponds to a change in the direction of BC. It was first estimated towards the right from B; it is now estimated towards the left from B.

Now, suppose BC be made greater than AB, then the point C will fall on the left of A.

And $m - n$, the expression for AC, will become negative. And we see that the expression for AC, which was positive by hypothesis when the distance was estimated on the right of A, has become negative when the distance is estimated in a contrary direction.

197. The question now arises, as to what interpretation is to be put upon a negative result when it appears in the solution of a problem.

Let it be required to find a number, which, when added to 6, will give a result equal to 4.* Then, from the conditions, we have the statement $x + 6 = 4$. Hence, $x = -2$. Now, the problem is purely arithmetical, and in arithmetic all numbers are regarded as positive. Therefore, the solution is absurd in an arithmetical sense, but it is true in an algebraic sense. For substituting -2 for x in the equation of the problem, we have $-2 + 6 = 4$, a true equation. The value satisfies the equation of the problem. It is, therefore, a true answer to the problem in an algebraic sense. But it is not a true answer to the problem as stated, for we were required to find a number, which, when *added* to 6, would give a *sum* equal to 4; and we have really found a number which, *subtracted* from 6, gave a difference equal to 4. The negative solution has then satisfied the equation of the problem, but has failed to fulfil the conditions as enunciated. The explanation of this difficulty is simple, when we return to the interpretation put upon a negative quantity. We have said that a negative quantity always indicates a change of direction or character. That change must be marked by a corresponding change of condition in the statement of a problem, and then the negative solution will be changed into a positive one. The preceding problem must be changed into this: required to find a number, which, when subtracted from 6, will give a difference equal to 4.

Then, $\qquad 6 - x = 4$, and $x = +2$.

A negative solution will not, then, satisfy a problem as enunciated, but will be the true answer to another problem in which there has been a change of condition corresponding to the indicated change of character. But negative solutions can be best explained by the discussion of the "Problem of the Couriers."

PROBLEM OF THE COURIERS.

198. Two couriers travel on the same road; the forward courier at the rate of b miles per hour, and the rear courier at the rate of a miles per hour. At 12 o'clock they are separated by a distance of m miles. The problem is, to determine how much time will elapse before they are together, and also the point of meeting.

Let the indefinite line

represent the road on which they are travelling; A, the position of the rear courier; B, of the forward courier at 12 o'clock; O, the unknown point of meeting, and x the required time of meeting. Then, from the conditions of the problem, $ax = \text{AO}$, $bx = \text{BO}$, and $\text{AB} = m$; and since, from the figure, we have, $\text{AO} - \text{BO} = \text{AB} = m$, we get the statement, $ax - bx = m$. Hence, $x = \frac{m}{a - b}$. The distance they are apart at 12 o'clock, divided by the difference of their rates of travel, will give the number of hours that must elapse, after 12, before they are together. The time, multiplied by the rate of travel, a, will give the distance, AO, from A to the unknown point of meeting O; or the time, multiplied by the rate b, will give the distance, BO. If, for instance, they are separated by a distance of 48 miles at 12 o'clock, and the forward courier travels at the rate of 4 miles per hour, and the rear courier at the rate of 6 miles per hour, then, $m = 48$, $a = 6$, and $b = 4$. Hence, $x = \frac{48}{6 - 4} = 24$ hours. $ax = 6 \,.\, 24 = 144$ miles $= \text{AO}$. $bx = 4 \,.\, 24 = 96$ miles $= \text{BO}$. Verification, $\text{AB} + \text{BO} = 48 + 96 = 144 = \text{AO}$.

We might make AO the unknown quantity, and call it y; then, $\text{BO} = y - m$. Then, $\frac{y}{a}$, the distance the rear courier will have to travel, divided by his rate of travel, will give the time that must elapse before he will overtake the forward courier. So, $\frac{y - m}{b}$ will give the time that the rear courier will be travelling before he is overtaken. And, since these are but different expressions for the same time, we get the equation $\frac{y - m}{b} = \frac{y}{a}$. Hence, $y = \frac{am}{a - b}$; and supposing, as before, that

$a = 6$, $b = 4$, and $m = 48$, we have $y = \text{AO} = \frac{6 \,.\, 48}{6-4} = 144$ miles. Then, $\frac{y}{a} = \frac{144}{6} = 24$ hours, as before.

199. We will confine our discussion mainly to the expression, $x = \frac{m}{a-b}$, in which x represents the unknown time. Since the value of a fraction increases as its denominator decreases, it is evident that, by making the difference between a and b very small, we can make the time very great. This is apparent, too, from the problem; for, if the rear courier travel but little faster than the forward, he will gain but little upon him. Now, if the denominator be made the smallest possible, the fraction will be the greatest possible. Hence, when the denominator is zero, which results from making $b = a$, the fraction must have the greatest possible value; or, in other words, an infinite value. Therefore, $x = \frac{m}{o} =$ infinity, symbolized by the character ∞. It will then be an infinite time after 12 o'clock before the couriers will be together; that is, they will never be together. This is plain, from the nature of the problem; for, if the couriers are separated by a distance of m miles at 12 o'clock, and travel at equal rates after 12, they must always keep at the same distance of m miles apart.

The character ∞, then, indicates impossibility. Whenever it appears, by examining the equation of the problem, we will discover an absurdity arising from the hypothesis that gave the infinite result. In this case, infinity was the result of the hypothesis, $a = b$. When $a = b$, ax is equal to bx, or $\text{AO} = \text{BO}$: a part equal to the whole, which is absurd.

The character ∞, then, indicates two distinct things, viz.: impossibility in the fulfilment of the required conditions, and absurdity in the conditions themselves.

200. Now, let b become greater than a, the denominator of the value of x becomes negative, and the whole fraction negative. What does this solution mean?

The sign of the time being minus, indicates that it refers to past time, because in the statement we regarded future time as positive. It is plain, also, from the nature of the problem, that the time of the couriers' being together, if together at all, must be past, not future. For the forward courier travelling more rapidly than the rear courier, will never be overtaken by him. The solution, then, fails to apply to the problem as enunciated; but it is a true solution for another question, viz.: how long before 12 o'clock were the two couriers together? Let us return to the particular problem: making $b = 6$ instead of $a = 6$. Then, $m = 48$, $a = 4$, $b = 6$, and $x = \frac{m}{a - b} = -24$. Now, it is obvious, that the forward courier, gaining two miles per hour on the rear courier, must have been with him 24 hours before 12, otherwise the two would not be separated by 48 miles at 12.

Returning to the equation in distance, $y = \frac{am}{a - b}$, we see that the distance, AO, also becomes negative under the hypothesis of $a < b$. This solution indicates impossibility for a point O in advance of B, because, when $a < b$, ax is also $< bx$, or AO $<$ BO, which is absurd. But it will be a true solution for a point O in rear of A.

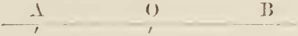
Because, for such a point, the relation $ax < bx$ will be true.

Resuming the values $m = 48$, $a = 4$, and $b = 6$, we get $y =$ AO $= -96$ miles; which agrees with the solution above, for the courier A, 24 hours before 12 o'clock, being at O, must be 96 miles from O at 12.

When the unknown quantity was time, we have seen that the negative solution indicated impossibility for future time, but gave a true answer to the question in past time. When the unknown quantity was distance, the negative solution indicated impossibility for forward distance, but gave a true answer to the question in distance estimated in the opposite direction. In general, a negative solution indicates that the problem, as enunciated, cannot be solved, but gives a true solution when the unknown quantity is changed in character or direction.

That being the case, a negative solution may always be changed into a positive solution by changing the character or direction of the quantity that produced the negative result. The negative solution, $x = \frac{m}{a - b}$, may be changed into a positive solution in three ways. First, by changing the direction of the time; that is, by changing $+ x$ into $- x$, which is equivalent to asking, how long before 12 o'clock the two couriers were together. Then the equation, $ax - bx = m$, becomes

$bx - ax = m$, or $x = \frac{m}{b - a}$, a positive result, because b is greater than a. Second, by changing $-b$ into $+b$, which is equivalent to turning the forward courier around to meet the rear courier. The equation then becomes $x = \frac{m}{a + b}$. The diagram corresponds to this; for, in this case, the point O will be between A and B, and we will have $ax + bx = AO + BO = AB$.



Third, by changing $+a$ into $-a$, and $-b$ into $+b$, which is equivalent to turning both couriers around, and making the courier B go in pursuit of the courier A, the equation becomes $-ax + bx = m$, or $x = \frac{m}{b - a}$, a positive result. It will be seen that the results in the first and third cases are the same, as they obviously ought to be.

201. Now, let $m = 0$, and $a > b$. Then, $x = \frac{0}{a - b} = 0$. The quotient of zero by any finite quantity is plainly zero; for the numerator of a fraction indicates the amount to be divided, and the division of nothing must give nothing. Thus, if a man has zero dollars to divide among three persons, the share of each will plainly be nothing. In the present instance, the solution, 0, shows that the couriers will be together at a zero time after 12; in other words, at 12 itself. This ought to be so; for, when $m = 0$, they are together at 12; and, since they travel at unequal rates, they will be together no more.

202. Now, let $m = 0$, and $a = b$. Then, $x = \frac{0}{0}$. To explain this symbol, it is necessary to notice particularly the conditions. The condition, $m = 0$, places the two couriers together; the condition, $a = b$, makes them travel at equal rates. Being together at 12, and travelling at equal rates, they must always be together. They will then be together at 1, at 2, 3, 3½, &c. There is, then, no *determinate* time at which they are together. Hence, $\frac{0}{0}$ is called the *symbol* of *indetermination*. It does not indicate that no solution can be found, but, on the contrary, that too many can be determined; and the indetermination consists in this, that any one of the infinite solutions will answer just

as well as any other. Suppose, for instance, the answer to the question, who discovered America? was, an inhabitant of Europe some time after the Christian era. The answer would be indeterminate, because equally applicable to countless millions.

By clearing the equation, $x = \frac{0}{0}$, of fractions, we get $0 . x = 0$. This will be a true equation, when $x = 1, 10, 1000$, anything whatever. Hence, we say that x is indeterminate. The diagram, also, shows the same thing.

For, when $m = 0$, the points A and B become the same; and, since $a = b$, then, AO = BO in all positions of the point O.

Thus, it is shown in three ways that $\frac{0}{0}$ is the symbol of indetermination. The symbol also arises from an identical equation; for, when $a = b$, and $m = 0$, we have $ax = ax$, an identical equation, in which x may have any value whatever.

By recurring to the equation in y, we will observe that the distance, from the common point of starting to the unknown point of meeting, also becomes indeterminate when $m = 0$ and $a = b$. This, obviously, ought to be so.

203. There is one remarkable exception to the foregoing statement, that $\frac{0}{0}$ is the symbol of indetermination. It is in the case of vanishing fractions. A vanishing fraction is one, which becomes $\frac{0}{0}$, in consequence of the existence of a common factor to the numerator and denominator, which has become zero by a particular hypothesis made upon it.

204. Take the expression $\frac{m - n}{m^2 - n^2} = \frac{0}{0}$, when $m = n$, but, by decomposing the denominator into its factors, we have $\frac{m - n}{(m - n)(m + n)}$. We see that $\frac{0}{0}$ is caused by the common factor, $m - n$, becoming 0 by the hypothesis $m = n$. Divide out this factor, and we have left $\frac{1}{m + n} = \frac{1}{2n}$, when $m = n$.

Again, take the fraction, $\frac{m^2 - n^2}{m - n} = \frac{(m+n)(m-n)}{m-n} = \frac{0}{0}$, when $m = n$. But, divide out $m - n$, and we have $\frac{m+n}{1} = 2n$, when $m = n$. So, $\frac{(m-n)^2}{m-n} = \frac{(m-n)(m-n)}{m-n} = \frac{0}{0}$, when $m = n$. But by division, the fraction becomes $\frac{m-n}{1} = 0$, when $m = n$. So. $\frac{m-n}{(m-n)^2} = \frac{m-n}{(m-n)(m-n)} = \frac{0}{0}$, when $m = n$. But, by division, we get $\frac{1}{m-n} = \frac{1}{0} = \infty$, when $m = n$.

We conclude, that a vanishing fraction has one of three true values; that it may be either finite, zero, or infinity. How, then, are we to decide whether the symbol $\frac{0}{0}$ indicates indetermination, or a vanishing fraction?

If the particular hypothesis which gives the symbol, does not make the given equation an identical equation, we may be certain that $\frac{0}{0}$ points out a vanishing fraction.

We will resume the subject of vanishing fractions more at length hereafter.

205. The foregoing symbols are of the highest importance, and ought to be remembered.

Arrangement in tabular form will assist the memory.

$$x = \frac{m}{0} = \infty;$$

that is, a finite quantity, divided by zero, equal to infinity. The symbol, ∞, indicates impossibility in the fulfilment, and absurdity in the conditions of the problem.

$$x = \frac{0}{A} = 0;$$

that is, zero, divided by any finite quantity, equal to zero. This is a true solution, unless it conflict with the condition of the problem.

$$x = -A.$$

The negative solution indicates that the problem, as enunciated, cannot be solved, but the negative result will be a true answer when the unknown quantity is changed in character or direction.

$$x = \frac{0}{0},$$

the symbol of indetermination, when there is no common factor in the numerator and denominator. By indetermination is meant, that there is an infinite system of values that will satisfy the conditions of the problem.

GENERAL PROBLEMS.

206. 1. A person employed a workman for m days, upon condition that the workman should receive n dollars every day that he worked, and should forfeit p dollars every day that he was idle; at the end of the time he received c dollars. How many days did he work, and how many was he idle?

Let $x =$ working days, then $m - x =$ idle days; and by the conditions, we get $nx - (m - x)\, p = c$.

Hence, $x = \dfrac{c + mp}{n + p}$, $m - x =$ idle days $= m - \dfrac{c + mp}{n + p} = \dfrac{nm - c}{n + p}$.

When will the last expression be zero, and when negative, and what do these solutions indicate?

The negative solution needs some explanation: c, the entire amount received, cannot exceed nm, what was paid for working m days, unless the workman had not only no forfeiture to pay, but also worked more than the time for which he was employed. The idle days have then changed their character, and become working days beyond the period for which the man's services were engaged.

2. The same problem as before, except the workman, at the end of the time, was in debt c dollars.

Ans. $x = \dfrac{mp - c}{n + m}$, and $m - x = \dfrac{nm + c}{n + p}$.

When will the first solution become zero, and when negative?

3. The same problem, except that the workman received nothing.

Ans. $x = \dfrac{mp}{n + p}$, and $m - x = \dfrac{nm}{n + p}$.

The above results are formulas, and may be made applicable to particular problems of the same kind.

4. In a certain college, the maximum mark for a recitation is 100.

A Freshman recited 5 times in mathematics; two days he got 95 for each recitation, and his average for the 5 days was 86. What did he average each of the other three days? *Ans.* 80.

5. Divide the number a into two such parts that the product of the first by m shall be equal to the quotient arising from dividing the second by n.

$$Ans.\ \frac{a}{1+nm},\ \text{and}\ \frac{anm}{1+nm}.$$

What effect has increasing n upon both results? Under what form must the second expression be placed to show that it is directly proportional to n and m? When will the two expressions be equal?

6. A father is 32 years old, and his son 8 years. How long will it be until the age of the father is just double that of the son?

Ans. 16 years.

7. A father is a years, and his son b years old. How long until the age of the father will be c times as great as that of the son?

$$Ans.\ x = \frac{a - cb}{c - 1}.$$

What do these values become when $c = 1$? What, when $cb = a$? What, when $cb > a$? What do these different solutions indicate? When $c = 1$, $x = \infty$, the symbol of absurdity, as it ought to be. The hypothesis makes the father and son equal in age after the lapse of b years. When $cb = a$, $x = 0$. A true solution, since the father is now c times as old as his son. When $cb > a$, x is negative. Future time being positive, past time is negative. The solution, then, indicates that $\frac{a - cb}{c - 1}$ years ago, the father was c times as old as his son.

8. A father is 64, and his son 16. How long will it be until the age of the father is 9 times as great as that of the son.

Ans. $x = -10$.

The solution indicates that 10 years ago the age of the father was 9 times as great as that of the son. The father was then 54, and the son 6 years old.

9. Bronze cannon (commonly, but improperly, called brass cannon) are composed of 90 parts of copper, and 10 parts of tin: $8\frac{9}{10}$ lbs. of copper lose 1 lb. when immersed in water, and $7\frac{3}{10}$ lbs. of tin lose 1 lb. when immersed in water. The Ordnance Board suspecting that some bronze cannon did not contain copper enough, immersed one of them,

weighing 900 lbs., and found its loss of weight to be $103\frac{3809}{6497}$ lbs. What was its composition?

Ans. 80 parts of copper, and 10 parts of tin.

Verify the results. The copper lost $89\frac{79}{89}$ lbs., and the tin $13\frac{51}{73}$ lbs. The sum of which is $103\frac{3809}{6497}$ lbs.

10. A fox, pursued by a greyhound, is 125 of his own leaps ahead of the greyhound, and makes 6 leaps to the greyhound's 5, but 2 of the greyhound's leaps are equal to 3 leaps of the fox. How many leaps will the fox make before he is caught by the greyhound?

Let $x =$ number of leaps made by the fox; then, since the greyhound makes but 5 leaps while the fox makes 6, he will make $\frac{5}{6}$ of a leap to one leap of the fox. Therefore, he will make $\frac{5}{6}x$ leaps while the fox makes x leaps. But each of the greyhound's leaps are equal to $\frac{3}{2}$ of a leap of the fox. Hence, the $\frac{5}{6}x$ leaps of the greyhound are equivalent to $\frac{3}{2} \cdot \frac{5}{6}x$ leaps of the fox; and since the greyhound has not only to run over the ground passed by the fox in making x leaps, but also that passed in making 125 leaps, we have the equation of the problem. $\frac{3}{2} \cdot \frac{5}{6}x = x + 125$. *Ans.* $x = 500$ leaps.

Verification. While the fox made 500 leaps, the greyhound made $416\frac{2}{3}$ leaps; and these were equal to $\frac{3}{2} \cdot 416\frac{2}{3}$, or 625 leaps of the fox, the entire number of leaps made by the fox.

Remarks.

This problem shows the importance of making the two members express the same thing by referring them to the same unit. The leaps of the greyhound have been expressed in terms of those of the fox. We might have made the unknown quantity represent the number of leaps made by the greyhound, and, in that case, the leaps of the fox must have been expressed in terms of those of the greyhound.

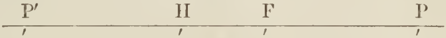
11. A fox, pursued by a greyhound, has a start of a leaps, and makes b leaps while the greyhound makes c leaps; but d leaps of the greyhound are equal to e leaps of the fox. How many leaps will the fox make before he is overtaken by the greyhound?

$$Ans.\ x = \frac{adb}{ec - db}.$$

When will $x = o$? When will it be equal to infinity? When will it be negative? How are these solutions explained?

x will be zero, when a is zero. x will be infinite when $ec = db$;

that is, when the number of leaps made by the greyhound, multiplied by the value of each leap, is equal to the number of leaps of the fox, multiplied by the value of each of his leaps. In that case, the hound will evidently never overtake the fox; and the solution indicates impossibility or absurdity. x will be negative when db is greater than ec; then the fox is running faster than the hound, and the distance represented by the x leaps must be estimated in a contrary direction.



Let H be the position of the hound; F, that of the fox; and P the point where the fox is overtaken by the hound. Then, when x is negative, we understand either, that the fox pursued the hound and caught up to him at some point, P′ on the left, or that, at some time previous to the fox being at F, and the hound at H, they were both together at P′, and the fox running faster than his pursuer gained upon him the distance HF $= a$.

12. A man, desirous of giving 4 cents apiece to some beggars, found that he had not money enough by 5 cents; he therefore gave them 2 cents apiece, and had 15 cents left. How many beggars were there, and how much money had he? *Ans.* 10 beggars: 35 cents.

13. A man, desirous of giving a cents apiece to some beggars, found that he had not money enough by b cents; he therefore gave them c cents apiece, and had d cents left. How many beggars were there, and how much money had the man?

Ans. $\frac{d+b}{a-c}$ beggars, and $\frac{ad+cb}{a-c}$ cents.

What values must a, b, c, and d have to make this problem the same as the last?

When $c > a$, the solution is negative; and negative solutions indicate a change of direction and character. The beggars become givers: the amount given becomes the amount received: the deficiency becomes a surplus, and the surplus a deficiency. When $c = a$, both solutions become infinite, and the symbol, ∞, here plainly indicates absurdity.

When d and b are both zero, the two solutions will be both 0, or $\frac{0}{0}$. Why?

14. A man, desirous of giving 2 cents apiece to some beggars, found that he had not money enough by 15 cents; he therefore gave them

4 cents apiece, and had 5 cents left. How many beggars were there, and how much money had the man?

Ans. — 10 beggars, and — 35 cents.

How is the solution explained? There were 10 givers, who each gave the man 2 cents, and 15 cents over; or each 4 cents apiece, lacking 5 cents in all.

15. A piratical vessel sails at the rate of r miles per hour for a hours, when she sustains some injury, and can only sail r' miles per hour. At the moment in which the piratical vessel is disabled, a sloop-of-war starts in pursuit, and sails at the rate of r miles per hour, from the point where the pirate first started. How long before the sloop will overtake the pirate?

Ans. $x = \frac{ar}{r - r'}$ hours.

When will this solution be zero? When negative? When infinite? Under what form must the fraction be placed to show that x is reciprocally proportional to r?

Make the general solution applicable to a particular example.

16. Divide a number, a, into two parts, which shall be to each other as m is to n.

Ans. $\frac{ma}{m + n}$, and $\frac{na}{m + n}$.

17. Divide a number, a, into three parts; such, that the first shall be to the second as n to m, and the second to the third as q to p.

Ans. $\frac{anq}{mp + mq + nq}$, $\frac{amq}{mp + mq + nq}$, $\frac{amp}{mp + mq + nq}$.

What supposition will make all the parts zero? What one of them? What two of them?

18. Divide a number, a, into four parts; such, that the first shall be to the second as n to m, the second to the third as q to p, and the third to the fourth as r to s.

Ans. $\frac{anqr}{nqr + mqr + mpr + mps}$, $\frac{amqr}{nqr + mqr + mpr + mps}$,
$\frac{ampr}{nqr + mqr + mpr + mps}$, $\frac{amps}{nqr + mqr + mpr + mp}$.

19. Milk sells in the City of New York at 4 cents per quart. A milkman mixed some water with 50 gallons of milk, and sold the mixture at 3 cents per quart without sustaining any loss by the sale. How much water did he put in the milk? *Ans.* $66\frac{2}{3}$ quarts.

20. Milk sells in Boston at a cents per quart. A milkman mixed a certain quantity of water with b quarts of milk, and sold the whole at c cents per quart without losing anything by the sale. How much water was added to the milk?

Ans. $x = \frac{ab - bc}{c}$ quarts.

The value of x is zero when $c = a$. In that case, evidently, no water is added. The value is infinite when $c = o$. Then the milkman gives away, gratuitously, an infinite quantity of water. When $c > b$, x is negative. Then we understand that a certain quantity of water is separated from, not added to the milk, and that the price of the milk is c cents per quart, and of the mixture a cents per quart.

21. How often are the hour and minute hands of a clock together? *Ans.* Every $65\frac{5}{11}$ minutes.

22. How often are the minute and second hands of a clock together? *Ans.* Every $1\frac{1}{59}$ minutes.

23. How often are the hour, minute, and second hands of a clock together? *Ans.* Every 720 minutes.

The last problem is solved by means of the least common multiple.

24. There is an island 60 miles in circumference. Three persons start from the same point to travel around it, travelling at the respective rates of 4, 6, and 16 miles per hour. How often will all three be together? *Ans.* Every 30 hours.

This problem is solved by means of the least common multiple.

25. There is an island a miles in circumference, around which three persons start to travel, at the rates of b, c, and d miles per hour. When will they all be together again?

Ans. In a hours, divided by the greatest common divisor of $c - b$, and $d - c$.

26. Same problem as 24, except that there are 4 persons travelling, at the respective rates of 3, 6, 12, and 27 miles per hour.

Ans. Every 20 hours.

27. A farmer purchases a tract of land for \$500, on a credit of 10 months, or \$480 cash. What is the rate of interest that makes these sums equivalent. *Ans.* 5 per cent.

Let x = interest upon \$100 for one month. The statement will be $100 + 10x : 100 :: 500 : 480$.

28. A farmer buys a tract of land for a dollars, payable in b months, or for c dollars cash. What is the rate of interest?

Ans. $x = \dfrac{100\,(a - c)}{cb}$.

What supposition will make this value zero? What two suppositions will make it infinite? What supposition will make it negative? and how is the negative solution explained?

29. Two numbers are to each other as 8 to 3; but if 8 be added to both numbers, the first will only be double the second. What are the numbers? *Ans.* 32 and 12.

30. Two numbers are to each other as a to b; but if c be added to both of them, the first will only be d times as great as the second. What are the numbers?

Ans. First, $\dfrac{ac\,(d-1)}{a-bd}$; second $\dfrac{bc\,(d-1)}{a-bd}$.

What do these solutions become when $d = 1$? What, when $bd = a$? What, when $bd > a$? What, when $d = 1$, and $bd = a$?

The first hypothesis gives a true solution. The second gives an absurd solution, as it ought, since bd can only equal a when the first number, after the addition of c to both numbers, exceeds the second proportionally as much as before. But this is impossible, since the smaller increases most rapidly. See Article 96.

The hypothesis, $bd > a$, gives a solution impossible for arithmetical quantities, but possible for algebraic. We must either suppose that two numbers are to be found, which result from the subtraction from a number not expressed, or we must change the character of the problem, and make c subtractive. When $bd = a$, and $d = 1$, the two numbers are represented by $\frac{0}{0}$, the symbol of indetermination. An infinite, or indeterminate, number of quantities will satisfy the conditions of the problem.

The following problems will illustrate the foregoing cases:

31. Two numbers are to each other as 4 to 3; but when 5 is added to both, the numbers are equal. That is, $d = 1$.

Ans. Both numbers zero.

32. The same as last, except that, after the addition, the first will be $\frac{4}{3}$ greater than the second. That is, $bd = a$.

Ans. Both infinite.

33. The same as 31, except that, after the additions, the first will be twice as great as the second. That is, $bd > a$.

Ans. -10, and $-7\frac{1}{2}$.

Change the character of the problem, and take 5 from both numbers, and the solutions will be $+10$ and $+7$.

34. The same as 31, except that the numbers were equal before the addition of 5 to both.

Ans. Both indeterminate, $\frac{0}{0}$. Any numbers will answer.

35. A father divided his estate, worth $1200, among his three sons, so that the share of the first should be to that of the second as 6 to 4; and so that the share of the third should be the greatest common divisor of the shares of the first and second. What will be the share of each? *Ans.* $600, $400, and $200.

36. A father divides his estate among his three sons, so that the share of the first shall be to that of the second as $\frac{mp}{q}$ is to $\frac{np}{s}$; and so that the share of the third shall be the greatest common divisor of the other two. The father's estate is worth a dollars: what is the share of each son?

Ans. First, $\frac{ams}{ms+nq+1}$; second, $\frac{anq}{ms+nq+1}$; third, $\frac{a}{ms+nq+1}$.

When will the shares of the first and second be equal? What will the share of the third be then? What will be the effect upon the share of the first to make $\frac{np}{s} = \infty$?

37. The difference of two numbers is a, and the difference of their squares is zero. What are the numbers?

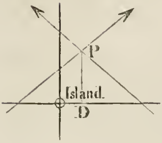
Ans. $\frac{a}{2}, -\frac{a}{2}$.

38. The sum of two numbers is $2a$, and the difference of their squares equal to c. What are the numbers?

Ans. $\frac{4a^2+c}{4a}$, and $\frac{4a^2-c}{4a}$.

What do these solutions become when $c = 0$? What, when $c = 4a^2$? What, when $c > 4a^2$?

39. Two ships sail in such a manner that the track of one cuts the parallel of latitude through the Island of St. Helena 40 miles on the west of that island, and cuts the meridian through the island 40 miles north of it; and that the track of the other cuts the parallel of latitude 80 miles on the east, and the meridian 80 miles on the north? Where will the two tracks intersect each other?



Ans. 20 miles east, and 60 miles north of the island.

Call DO, x; then, $PD = x + 40$; and $PD = 80 - x$. Hence, $40 + x = 80 - x$; then, $x = 20$; and $PD = 60$.

40. A Yankee mixes a certain number of wooden nutmegs, which cost him $\frac{1}{4}$ cent apiece, with a quantity of real nutmegs, worth 4 cents apiece, and sells the whole assortment for \$44; and gains \$3·75 by the fraud. How many wooden nutmegs were there?

Ans. 100.

41. A Yankee mixed a certain quantity of wooden nutmegs, which cost him $\frac{1^{\text{th}}}{b}$ part of a cent apiece, with real nutmegs, worth c cents apiece, and sold the whole for a dollars. He gained d cents by the fraud. How many wooden nutmegs were there?

Ans. $x = \dfrac{db}{bc - 1}$.

What does this become when $d = 0$? What, when $b = 0$? What, when $bc = 1$? What, when $bc < 1$.

42. The sum of three numbers is 200; the first is to the second as 5 to 4, and the third is the greatest common divisor of the first two. What are the numbers? *Ans.* 100, 80, and 20.

43. The sum of three numbers is a; the first is to the second as b to c, and the third is the least common multiple of the first two. What are the numbers?

Ans. $\dfrac{ab}{b + c + bc}$, $\dfrac{ac}{b + c + bc}$, $\dfrac{abc}{b + c + bc}$.

What single hypothesis will make them all equal? What will these values become when $a = 115$, $b = 25$, and $c = 15$.

44. A northern railroad company is assessed \$120,000 damages for contusions and broken limbs, caused by a collision of cars. They

pay \$5000 for each contusion, and \$6000 for each broken limb; and the entire amount paid for bruises and fractures is the same. How many persons received contusions, and how many had their limbs broken? *Ans.* 12 of the former, and 10 of the latter.

45. Same problem as the last, except representing the assessment by a, the price of each contusion by c, and that of each broken limb by b.

Ans. $\frac{a}{2c}$, and $\frac{a}{2b}$.

What do these values become when $c = 0$? What, when $b = 0$? What, when $a = 0$? What, when $b = c$?

46. The reservoir at Lexington contains 48,000 gallons of water, and supplies the town and Virginia Military Institute. If all the conducting pipes were closed, the reservoir would supply the town and the Institute for $57\frac{3}{5}$ hours, and the town alone for 96 hours. How many gallons does the Institute use per hour.

Ans. $333\frac{1}{3}$ gallons.

47. Two pipes will exhaust a cistern containing a quantity of water, represented by q, in a hours, and the first will, alone, exhaust it in b hours. How long will it take the second pipe to empty it, and how much does it exhaust per hour.

Ans. $\frac{ab}{b-a}$ hours, and $\frac{q(b-a)}{ab}$ gallons per hour.

What do these solutions become when $a = b$, and $a > b$?

The negative solution can be illustrated by an example.

48. Two pipes, a and b, will exhaust a cistern in 4 hours; and the pipe a can, alone, exhaust it in 2 hours. In what time can the pipe b empty the cistern? *Ans.* In -2 hours.

The solution being negative, may refer to past time, and indicate that, during the two hours before the pipe a began to play, the pipe b had exhausted the cistern. The two hours play of the pipe b, added to the two of the pipe a, give the 4 hours of the problem. Or, we may suppose that the character of the pipe b has been changed, and that it has been a supplying pipe for two hours, and, consequently, the pipe a has been twice the time in exhausting the cistern.

49. A man asks \$120 cash for his horse, or \$126·30 on a credit of 9 months. Supposing these valuations to be equal, what is the rate of interest? *Ans.* 7 per cent.

50. A man values his horse at c dollars cash, or b dollars on a credit of a months. What is the rate of interest?

Ans. $x = \frac{1200\,(b - c)}{ac}$.

What does x become when $b = c$? What, when $a > b$? What, when a, or $c = 0$?

51. There is an island 32 miles in circumference: two persons start to travel around it, the first at the rate of 11 miles per day, and the second at the rate of 3 miles per day. When will the distance between them be equal, estimated in either direction around the island?

Ans. At the end of the second day.

52. Two persons start to travel around an island a miles in circumference, the first at the rate of b miles per day, and the second at the rate of c miles per day. When will the distances between them be equal, estimated in both directions around the island?

Ans. $x = \frac{a}{2\,(b - c)}$

What will this solution become if the travellers move at equal rates? What, if the second travels the fastest? What, if the second stops?

53. Three persons start to travel around an island which is a perfect circle, 120 miles in circumference. The first travels at the rate of 4 miles per day, the second at the rate of 8 miles per day, and the third at the rate of 6 miles per day. How many days will elapse until the lines joining their respective positions, and the centre of the island, will trisect the circumference, under the supposition that the second, only, has gone past the starting point on a second tour of the island?

Ans. 20 days.

54. Three persons travel around a circular island, 160 miles in circumference. They start together, and travel at the respective rates of 7, 5, and 2 miles per hour. How long will it be until the third will be between the first and second, and equally distant from them, under the hypothesis that the first and second, only, have passed the starting point? *Ans.* 40 hours.

55. Three persons travel around an island a miles in circumference. They travel at the respective rates of b, c, and d miles per hour. How long will it be until the third will be half way between the first and second, and how far will the first have travelled?

Ans. Distance, $\frac{2ab}{b - 2d + c}$; time, $\frac{2a}{b - 2d + c}$.

What do these values become when $2d = b + c$? What, when $2d = c$? How do you explain these results?

56. Divide the number 24 into two such parts that their product be the greatest possible.

Let x express the excess of one of the parts over the half of 12, then, $6 + x$, and $6 - x$ will represent the two parts. And $(6 + x)(6 - x)$, $36 - x^2$ is to be the greatest possible. This result will evidently be the greatest possible when $x = 0$, or when the parts are equal.

57. Divide the quantity a into two such parts that their product shall be the greatest possible. What are the parts?

Ans. $\frac{a}{2}$, and $\frac{a}{2}$.

58. Two men have each an end of a pole 6 feet long upon their shoulders, with a burden suspended from it. The share of the load sustained by each man is inversely proportional to the distance of the load from his shoulder. The proportion of the weight borne by one man is to that borne by the other as 3 to 2. How far is the burden from the shoulder of the first man? *Ans.* $2\frac{2}{5}$ feet.

59. Same problem as the last, except the representation of the length of the pole by a, and the proportion by b and c. How far is the burden from the shoulder of each man?

Ans. $\frac{ac}{b + c}$ feet, and $\frac{ab}{b + c}$ feet.

When will these distances be equal? When one double the other?

60. A general wishing to range his men in a solid square, found that he had too many men by 100. He increased each side of the square by one man, and then found that he had but 39 too many How many men had he? *Ans.* 1000.

Let x = side of the square.

61. A general ranged his army in a solid square, and found that he had a more men than would enter the square. He then increased each side of the square by one man, and found that he had b more men than could enter the square. How many men were in each side of the square?

Ans. $x = \frac{a - (b + 1)}{2}$.

What does this value become when $b + 1 = a$? What, when $b = -1$? How do you explain these results? The last result will confirm a truth hereafter to be demonstrated.

62. Divide the number 20 into three such parts that the half of the first, the one-third of the second, and the one fifth of the third shall be equal. What are the parts? *Ans.* 4, 6, and 10.

For, let x represent the equal result after the division of the three parts by 2, 3, and 5; the parts themselves will plainly be represented by $2x$, $3x$, and $5x$. Hence, $2x + 3x + 5x = 20$, or $x = 2$. Then, $2x = 4$, $3x = 6$, and $5x = 10$.

63. Divide the number a into three such parts that the first divided by b, the second by c, and the third by d, shall all be equal. What are the parts?

Ans. $\frac{ab}{b + c + d}$, $\frac{ac}{b + c + d}$, and $\frac{ad}{b + c + d}$.

When will the three parts be equal? What will be the effect of making $b = 0$? How do you explain the result?

64. Two laborers are engaged to dig a ditch; the first can dig it in 5 days, the second in $3\frac{1}{3}$ days. How long will it take the two, working together, to dig the ditch? *Ans.* 2 days.

Verify the result.

65. Two men are employed to dig a ditch; the first, alone, can dig it in a days, and the second, alone, can dig it in b days. How long will it take the two, laboring together, to perform the work?

Ans. $x = \frac{ab}{b + a}$.

When will the time be one half of that required by the first to dig it? What change will this solution undergo when the second man fills up instead of digs out? What will be the expression for the time in that case, when he fills up as fast as the first digs out?

This problem shows clearly that a change in a condition will be accompanied by a change of sign.

66. A debt of $150 was paid in dollar and five-cent pieces. The dollar and five-cent pieces were 1100 in number. How many dollar, and how many five-cent pieces were used?

Ans. 100 dollar pieces, and 1000 five-cent pieces.

67. A debt of a cents was paid in b and c cent pieces, and the total

number of these pieces was equal to d? How many pieces of each kind were used.

Ans. $\dfrac{bd - a}{b - c}$, and $\dfrac{a - cd}{b - c}$.

What do these solutions become when $bd = a$, and $cd = a$? What, when $a > bd$, and $cd > a$? What, when $b = c$? Explain these results, especially the last.

68. The sum of three numbers is 420: the first is to the second as 6 to 7, and the third is just as much greater than the second as the second is greater than the first. What are the numbers?

Ans. 120, 140, and 160.

69. The sum of three numbers is a: the first is to the second as b to c, and the second exceeds the first as much as the third exceeds the second. What are the parts?

Ans $\dfrac{ac}{3b} \cdot \dfrac{a}{3}$ and $\dfrac{2ab - ac}{3b}$.

We observe that, when quantities bear the above relation, the mean is always the third of the whole.

What single hypothesis will make all three parts equal? What is the effect upon the three parts of making $c = 2b$? Explain this result?

70. At the Woman's Rights Convention, held at Syracuse, New York, composed of 150 delegates, the old maids, childless-wives, and bedlamites were to each other as the numbers 5, 7, and 3. How many were there of each class? *Ans.* 50, 70, and 30.

ELIMINATION BETWEEN SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

207. Simultaneous equations are those which can be satisfied for the same values of the same unknown quantities. Elimination can only be performed upon simultaneous equations. It is a process by which we deduce from two or more equations, containing two or more unknown quantities a single equation, containing one unknown quantity. The single equation so found is called the final equation, and since it contains but one unknown quantity does not lead to arbitrary values.

We will first take the simplest case, that of two equations involving two unknown quantities. The elimination of one of these unknown quantities may be effected in four ways.

1. By Comparison.
2. By Addition and Subtraction.
3. By Substitution.
4. By the Greatest Common Divisor.

ELIMINATION BY COMPARISON.

208. Take the equations $2y = 2x + 4$,
and $3y = 6x - 12$.

From the first we get $y = x + 2$,
and from the second, $y = 2x - 4$.

Now, since, by hypothesis, the y and x in one equation are equal to the y and x in the other equation, we can equate the values of y, and make the two x's represent the same thing. Hence, $x + 2 = 2x - 4$. From which we get $x = 6$. Solving the first equation with respect to x, we get $x = y - 2$; and from the second we get $x = \frac{y}{2} + 2$.

Now, since the x's are equal by hypothesis, we have a right to equate their values. Hence, $y - 2 = \frac{y}{2} + 2$.

Assuming that the y's are equal in the two members of this equation, we have $y = 8$. Hence, the values of x and y are $x = 6$, and $y = 8$. It will be seen that these values satisfy both equations.

Elimination by comparison between two equations with two unknown quantities consists in solving both equations with reference to one unknown quantity, and equating the values so found. This eliminates the first unknown quantity, and gives a single equation, from which the value of the second can be found. The second unknown quantity being eliminated in like manner, the value of the first can be found.

209. If there had been three unknown quantities, and but two equations, there would have been, after the elimination of y, a single equation with two unknown quantities, and the values of these two unknown quantities would, of course, have been arbitrary. If there had been two equations and but one unknown quantity, the equation could not, in general, be satisfied by the same value for that unknown quan-

tity. The equations $x + 2 = a$, and $x - 2 = a$, cannot be satisfied for the same values of x.

We see, then, that the number of equations must be precisely equal to the number of unknown quantities.

210. The equations have been combined under the supposition that they were simultaneous. If they are not so, the hypothesis has been absurd, and the result ought to indicate the absurdity. Take the manifestly absurd equations,

$$y = 2x + 2,$$

and
$$y = 2x - 2.$$

Combining, we get $2x - 2x$, or $0x = 4$. Hence, $x = \frac{4}{0} = \infty$.

The absurdity of the hypothesis is here pointed out by the symbol of absurdity.

211. If the equations are the same, or differ only in form, they can, obviously, be satisfied by an indeterminate number of values for one of the unknown quantities, provided that of the other is deduced after the substitution of the assumed value of the first.

Take,
$$y = 2x + 2,$$

and
$$y = 2x + 2.$$

Combining, we get $0y = 0$, or $y = \frac{0}{0}$. Which indicates that y may have any value whatever. Suppose we assume $y = 4$; this value for y, substituted in either of the equations, will give $x = 1$; and the two values, $y = 4$, and $x = 1$, will satisfy both equations. Assuming arbitrary values for y, we get an infinite number of values for x, and the solution becomes wholly indeterminate.

ELIMINATION BY ADDITION AND SUBTRACTION.

212. Resume the equations,

$$2y = 2x + 4,$$

and
$$3y = 6x - 12.$$

Multiply the first equation by 3, and the second in like manner by 2, and we will have

$$\begin{array}{r} 6y = 6x + 12 \\ 6y = 12x - 24 \\ \hline 0 = 6x - 36. \end{array}$$

If these equations are simultaneous, the $6y$ in the first equation is equal to the $6y$ in the second. And, since the x's will also be equal, the result of the subtraction of the equations, member by member, will be $0 = 6x - 36$, or $x = 6$, the same value as before found.

Now, to eliminate x in order to find the value of y, the first equation must be multiplied by 3, and the second must remain as it is.

We will get, $6y = 6x + 12,$

and $3y = 6x - 12,$

and by subtraction, $3y = 24$, or $y = 8$.

The same value as when we eliminated by comparison.

It will be seen that the coefficients of the unknown quantity to be eliminated have been made equal in the two equations; and, since they had like signs, the elimination could only be effected by subtraction. Had these coefficients, however, been affected with contrary signs, it would have been necessary to add the equations member by member.

Take as an illustration,

$$2y - x = 3$$

$$x - y = -1, \text{ by addition, } x \text{ is eliminated.}$$

$$y = 2$$

To eliminate y, multiply the second equation by 2, and we get

$$2y - x = 3$$

$$2x - 2y = -2,$$

and by addition, $x = 1$

The two values are then $y = 2$, and $x = 1$, and these satisfy both equations.

Elimination by Addition and Subtraction consists in making the coefficients of the unknown quantity to be eliminated the same in both equations, and then subtracting the equations member by member, if these coefficients have like signs, or adding them member by member if they have unlike signs.

It will be seen that the method of elimination by addition and subtraction involves no fractions, whilst the method by comparison, in general, does involve fractions. To eliminate by comparison between the equations

$$3y = x + 2,$$

and $2y = x - 2,$

will involve the fractions $\frac{x+2}{3}$, and $\frac{x-2}{2}$.

ELIMINATION BY SUBSTITUTION.

213. This consists in finding the value of one of the unknown quantities in one of the equations in terms of the other unknown quantity, and substituting the value found in the second equation, so as to determine the value of the second unknown quantity in known terms.

Resume the equations,

$$2y = 2x + 4,$$

and $$3y = 6x - 12.$$

From the first we get $y = x + 2$; and this value of y, substituted in the second equation (since the y's and x's are, by hypothesis, the same in both equations), gives $3x + 6 = 6x - 12$. Hence, $x = 6$. In like manner, x, found from the first equation, $x = y - 2$, substituted in the second, gives $3y = 6(y - 2) - 12$, or $y = 8$. The two values are then $x = 6$, and $y = 8$, as found by the other two methods.

This process will also, in general, involve fractions.

ELIMINATION BY THE GREATEST COMMON DIVISOR.

214. This process consists in transposing all the terms of the two equations to the first member, and then dividing the polynomials in the first member by each other, as in the method of finding the greatest common divisor, until a remainder is found which contains but one unknown quantity. This remainder, placed equal to zero, constitutes the final equation, and the value of one of the unknown quantities can be deduced from it. The value of the other unknown quantity can be found by a similar process.

Resume the equations,

$$2y = 2x + 4,$$

and $$3y = 6x - 12,$$

transposing, $$2y - 2x - 4 = 0,$$

and $$3y - 6x + 12 = 0.$$

Multiplying the second equation by 2, and dividing, we get

$$\begin{array}{l|l} 6y - 12x + 24 & 2y - 2x - 4 \\ \underline{6y - 6x \quad - 12} & \quad 3 \qquad \text{Quotient.} \\ \quad - 6x \quad - 36 = 0, \text{ or } x = 6. & \end{array}$$

To eliminate x, arrange with reference to x, and we have

$$\begin{array}{r|l} -6x + 3y + 12 & -2x + 2y - 4 \\ \underline{-6x + 6y - 12} & \quad +3 \qquad \text{Quotient.} \\ -3y + 24 = 0, \text{ or } y = 8 & \end{array}$$

It only remains to be shown the reason why the remainder is placed equal to zero rather than to 10, or anything else.

Let $A = 0$ represent the first equation after it has been prepared for division. Let $B = 0$ represent the second equation. Let Q represent the quotient after the division of A by B, then,

$$\begin{array}{l|l} A & B \\ \underline{BQ} & Q \\ A - QB = 0. & \end{array}$$

Now, since A is zero, and B zero, $A - QB$ is plainly zero.

For equations of the first degree there will be but one remainder. But if the equations be of a higher degree than the first, there will be two or more remainders. But the same course of reasoning will show that each successive remainder must be zero. For the first remainder having been shown to equal zero, and the divisor B also equal to zero, the second remainder must also be equal to zero. Let B' represent the second remainder, and Q' the quotient resulting from the division of B by B'.

Then,

$$\begin{array}{l|l} B & B' \\ \underline{Q'B'} & Q' \\ B - Q'B' = 0. & \end{array}$$

Since $B = 0$, and $B' = 0$, plainly $B - Q'B'$ must be zero.

The third remainder, the fourth, and so on, can, in like manner, be shown equal to zero.

215. If we combine equations, which are not simultaneous, by either of the first three methods, the absurdity of the hypothesis is shown by ∞, the symbol of absurdity in the result. But when we combine such equations by the fourth method, the absurdity appears in the final equation having no unknown quantity.

Take the equations

$$y = 2x + 2,$$

and

$$y = 2x + 4.$$

Combining by last process,

$$\begin{array}{l|l} y - 2x - 2 & y - 2x - 4 \\ \underline{y - 2x - 4} & \\ \quad\quad + 2 = 0, \text{ which is absurd.} & \end{array}$$

216. Though it is usual to say that the absurdity of combining equations which are not simultaneous is shown by the final equation, yet we might retain the trace of one of the unknown quantities, and then we would still have the symbol ∞. In the last example, we might retain the trace of y, and the final equation would be $0y + 2 = 0$; whence $y = \infty$.

Remarks.

217. Of the four methods of elimination, the last is generally used when the degree of the second equation is higher than the first, and the second method (by addition and subtraction) is preferable for simple equations, since it does not involve fractions. Elimination by substitution is generally associated with the other three methods after the value of one unknown quantity is found; this value is generally substituted in one of the given equations, and we are thus enabled to deduce that of the other.

Take the equations,

$$2y = 2x + 4,$$

and

$$3y = 6x - 12.$$

From the first we have, $y = x + 2$, and from the second $y = 2x - 4$. Equating the two values of y, we get $x + 2 = 2x - 4$. Hence, $x = 6$. This value for x, substituted in the first equation, gives $2y = 16$, or $y = 8$. And substituted in the second, gives $3y = 24$, or $y = 8$. Hence, the given equations are simultaneous. But if the substitution of the value for x in the two equations gave different values for y, we would conclude that the equations were not simultaneous.

218. *Examples in elimination between two simple equations of the first degree involving two unknown quantities.*

1. Find the values of x and y in the equations,

$$y = ax + b,$$

and

$$y = a'x + b'.$$

Ans. $x = -\dfrac{b - b'}{a - a'}$, and $y = \dfrac{ab' - a'b}{a - a'}$.

The hypothesis, $a = a'$, makes both values infinite. The combined equations show that when $a = a'$, $a'b$ must equal ab'; that is, unequal multiples of the same quantity must be equal, which is absurd. The equations, in fact, represent two straight lines, and the found values of x and y represent their point of meeting. The hypothesis, $a = a'$, makes the lines parallel, and their point of meeting is, of course, at an infinite distance.

Making $a = a'$ and $b = b'$, x and y both become $\frac{0}{0}$, or indeterminate. In this case, the two equations become identical, and can be satisfied by any values for x and y, as the symbol $\frac{0}{0}$ indicates. By this we mean that arbitrary values may be given to either x or y, and these arbitrary values for one of the unknown quantities, taken in connection with the deduced values of the other, will satisfy both equations.

When $a = a'$, and $b = b'$, the two lines coincide, and their point of meeting, being any where on the common line, is, of course, indeterminate.

When $b = b'$, we will have $x = 0$, and $y = b$.

2. Combine $2y + 3x = 4$,

and $y - 6x = 7$.

Ans. $x = -\frac{2}{3}$, and $y = 3$.

3. Combine $2y + 3x = 0$,

$y - 6x = 0$.

Ans. $x = 0$, and $y = 0$.

When there is no known term or terms in the two equations the values of x and y will always be zero, since these values will satisfy both equations.

4. Combine $2y + \frac{x}{2} - 4\frac{1}{2} = 0$.

and $3y + \frac{12x}{5} = 8\frac{2}{5}$.

Ans. $x = 1$, and $y = 2$.

5. Combine $\frac{x}{7} + \frac{y}{5} - 1 = 0$.

and $x + 2y + 4 = 0$.

Ans. $x = 32\frac{2}{3}$, $y = -18\frac{1}{3}$.

6. Combine $\frac{x}{4} + \frac{x}{8} + \frac{y}{5} + \frac{y}{15} = 11$.

and $x - \frac{x}{2} + \frac{x}{3} + y = 36\frac{2}{3}$.

Ans. $x = 8$, and $y = 30$.

7. Combine $3y - 3x + 6 = 0,$
and $7y - 7x = 14.$

Ans. $y = \infty$, and $x = \infty$.

8. Combine $\frac{a'y}{2} - \frac{7a'x}{2} + 2a' = 0.$

and $\frac{a'y}{3} = \frac{7a'x}{3} - \frac{4a'}{3}.$

Ans. $x = \frac{0}{0}$, and $y = \frac{0}{0}$.

9. Combine $\frac{1}{y} + \frac{1}{x} + 4 = 0.$

and $\frac{3}{y} - \frac{2}{x} - 2 = 0.$

Ans. $x = -\frac{5}{14}$, and $y = -\frac{5}{6}$.

Regard $\frac{1}{y}$, or $\frac{1}{x}$, as the quantity to be eliminated.

10. Combine $\frac{a}{y} + \frac{a}{x} - 4a = 0.$

and $\frac{3a}{y} - \frac{2a}{x} = 2a.$

When one of the unknown quantities is wanting, it may be written with a zero coefficient.

11. Combine $y = x + 2,$
and $y = 2$, or $y = 0x + 2.$

Ans. $x = 0$, and $y = 2$.

12. Combine $x = 0,$
and $y = 7x + 4.$

Ans. $x = 0$, and $y = 4$.

13. Combine $x + y = a$
and $ax + y = b.$

Ans. $x = \frac{b - a}{a - 1}$, and $y = \frac{a^2 - b}{a - 1}$.

What do these values become when $a = 1$? Why? What, when $b = a$? What, when $a = 0$? What, when $b = a^2$?

14. Combine $\frac{a}{x} + \frac{y}{b} = c.$

and $\frac{a}{x} + y = d.$

Ans. $x = \frac{a(b - 1)}{bc - d}$, and $y = \frac{(c - d)\,b}{1 - b}$.

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What do these values become when $b = 1$? What, when $c = d$? What, when $c = d$, and $b = 1$? What, when $b = 0$? How are the results explained?

15. Combine $$\frac{a}{x} + \frac{b}{y} = c.$$

and $$\frac{a'}{x} + \frac{b'}{y} = c'$$

Ans. $x = \dfrac{ab' - a'b}{cb' - bc'}$, and $y = \dfrac{ab' - a'b}{ac' - a'c}$.

Solve this example by the four methods of elimination. When x is eliminated by the last method, the final equation in y ought to be $(ac' - a'c)\,y + a'b - ab' = 0$.

What do the values of x and y become when $a'b = ab'$? Why? What do these values become when $bc' = cb'$? What, when $a = a'$, $b = b'$, and $c = c'$?

16. Combine $$y + \frac{a}{b}y + x = c,$$

and $$y - x = c.$$

Ans. $x = -\dfrac{ac}{2b + a}$ and $y = \dfrac{2bc}{2b + a}$.

The final equation in x, when y is eliminated, ought to be $(2b + a)\,x + ac = 0$.

What do these values become when $c = 0$? Why? What do they become when $a = 0$? What, when $b = 0$?

17. Combine $$\frac{y}{x} + y = 2a\,(1 + a).$$

and $$y + 2x - 2a = 2a^2.$$

Ans. $x = a$, and $y = 2a^2$.

Eliminating by substitution, we get $y = 2x^2$. Hence, the second equation will give $2x^2 - 2a^2 + 2x - 2a = 0$, or $2\,(x^2 - a^2) + 2\,(x - a) = 0$, or $2\,(x - a)\,(x + a + 1) = 0$. Dividing out by the factors $2\,(x + a + 1)$, we have $x - a = 0$, or $x = a$. The same result may be obtained by the fourth method of elimination.

18. Combine $$\frac{y}{x} + y = 2b,$$

and $$\frac{a}{x} + 2y = 2b + a.$$

Combining by fourth method, we have

$$(y-2b)\,x+y \mid (2y-2b-a)\,x+a.$$

Preparing for division by multiplying by

$$\begin{array}{l|l}
\multicolumn{1}{c}{(2y-2b-a)} & \\
\hline
x\,(y-2b)\,(2y-2b-a)+y\,(2y-2b-a) & \\
x\,(y-2b)\,(2y-2b-a)+ay-2ba & y-2b. \text{ Quotient.} \\
\hline
\end{array}$$

$$2y^2-2ay-2by+2ba=2y\,(y-a)-2b\,(y-a)=0. \text{ Remainder.}$$

or, $(y-a)\,(2y-2b)=0$, the final equation.

Divide out by $2y-2b$, and we have $y-a=0$, or $y=a$; and this value for y, substituted for either of the equations, gives $x=\frac{a}{2b-a}$. We might have divided out by $y-a$, and then we would have had $2y-2b=0$, or $y=b$; and this value for y would have given $x=1$. The equations, then, admit of two systems of values, $y=a$, and $x=\frac{a}{2b-a}$; and $y=b$, and $x=1$. Example 17 gives, also, a second system, $x=-(a+1)$, $y=2\,(a+1)^2$.

19. Combine $\frac{y}{x}+\frac{x}{y}=2,$

and $y-x=0.$

Ans. $x=\frac{0}{0}$, and $y=\frac{0}{0}$.

20. Combine $\frac{y}{x}+\frac{x}{y}-\frac{1}{2}=2$

and $y-x=2.$

Ans. $x=2$, and $y=4$, or $x=-4$, and $y=-2$.

Combining by substitution, we get $x^2+2x-8=0$; or, adding and subtracting unity, $x^2+2x+1-9=(x+1)^2-3^2=(x+1+3)\,(x+1-3)=0$.

By suppressing the first factor, we get $x=2$; and by suppressing the second, we get $x=-4$.

21. Combine $yx-x=6,$

and $x-y=-2.$

Ans. $x=2$, and $y=4$; or $x=-3$, and $y=-1$.

In this case, add and subtract $\frac{1}{4}$.

22. Combine $$yx - \frac{x}{y} = 0.$$
and $$y + x = 2.$$
Ans. $x = 1$, and $y = 1$; or $x = 3$, and $y = -1$.

23. Combine $$x + \frac{ax}{2} + y = 2,$$
and $$x + 2by = 0.$$
Ans. $x = \dfrac{-4b}{1 - b(a + 2)}$, and $y = \dfrac{2}{1 - b(a + 2)}$.

What do these values become when $a = -2$? What, when $b = 0$?

24. Combine $$x + b - a + \frac{y}{2} = \frac{a - b}{2},$$
and $$2x + \frac{2y}{a - b} = 2a - 2b + 2.$$
Ans. $x = a - b$, and $y = a - b$.

25. Combine $$y = ax + b,$$
and $$y = 2.$$
Ans. $x = \dfrac{2 - b}{a}$, and $y = 2$.

What do these values become when $a = 0$? What, when $b = 2$? Explain these results.

219. *Elimination between any number of simultaneous equations.*

The same principles govern the elimination of any number of simultaneous equations as have been shown to govern the elimination between two equations with two unknown quantities. No specific rules can be given for elimination, because each equation may contain all the unknown quantities; or, a part only of them may contain all. It may be even that no equation, or but one, contains all the unknown quantities. The main thing to be observed is, to eliminate the same unknown quantity from all the equations that contain it. We will then have one unknown quantity less than before, and one equation less. Continue the process of elimination, until a single equation with a single unknown quantity is obtained. If the number of unknown quantities is greater than the number of equations it will be impossible to obtain a single equation with one unknown quantity, because the number of equations reduced by elimination is always equal to the number of unknown quantities reduced. Two eliminations reduce the number of unknown quantities and equations by two; three eliminations by three, &c. It is evident, then, that when the number of unknown quantities exceed the number of equations, the last equation obtained will contain two or more unknown quantities, and will, consequently, be an indeterminate equation.

220. If the number of equations exceed the number of unknown quantities, it is plain that, before all the equations have been freed from their unknown quantities, we will get a single equation with but one unknown quantity. The value of this unknown quantity then can be determined, and it may not be such as to satisfy all the equations.

221. The number of equations and unknown quantities must not only be equal to each other, but the equations must be different in *character*, not in *form* merely. The equations $y = 2x + 2$, and $2y = 4x + 4$, differ only in form, and it is impossible to eliminate between them.

EXAMPLES.

1. Solve the three equations,

$$2y - x + z = 2,$$
$$2y + 2x + 4z = 8,$$
$$3y + 13x + 3z = 19.$$

Ans. $y = 1$, $x = 1$, and $z = 1$.

2. Solve the three equations,

$$y + x + z = 9,$$
$$y - x + z = 3,$$
$$y - x - z = -5.$$

Ans. $y = 2$, $x = 3$, and $z = 4$.

3. Solve the three equations,

$$y + x + z = a + b + c,$$
$$y - x + z + b - c = a,$$
$$y - 2x - 3z = a - 2b - 3c.$$

Ans. $y = a$, $x = b$, and $z = c$.

4. Solve the three equations,

$$x + 2y = 4,$$
$$x + z = a,$$
$$y - z = b.$$

Ans $y = 4 - (a + b)$, $x = 2(a + b) - 4$, and $z = 4 - (a + 2b)$.

5. Solve the three equations,

$$x - y = a - b,$$
$$x + 2 = a,$$
$$x + y + z - c = 4.$$

Ans. $y = b - 2$, $x = a - 2$, and $z = 8 + c - (a + b)$.

6. Solve the three equations,

$$\frac{y}{2} - x - \frac{a}{2} + 1 + \frac{z}{b} = \frac{c}{b}.$$
$$2ax + \frac{1}{2} - 2a + \frac{x}{2} + y - z = a - c + 1.$$
$$y - 2x + 3z = a + 3c - 2.$$

Ans. $y = a$, $x = 1$, and $z = c$.

7. Solve the three equations,

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0,$$
$$A''x + B''y + C''z + D'' = 0.$$

Ans. $$y = \frac{(A'C'' - A''C')D + (A''C - AC'')D' + (AC' - A'C)D''}{(A'B'' - A''B')C + (A''B - AB'')C' + (AB' - A'B)C''}$$

$$x = \frac{(B''C' - B'C'')D + (BC'' - B''C)D' + (B'C - BC')D''}{(A'B'' - A''B')C + (A''B - AB'')C' + (AB' - A'B)C''}$$

$$z = \frac{(A''B' - A'B'')D + (AB'' - A''B)D' + (A'B - AB')D'}{(A'B'' - A''B')C + (A''B - AB'')C' + (AB' - A'B)C''}$$

What do these become when D, D′, and D″ are all equal to zero? Why? What do they become when either A, A′, A″, or B, B′, B″, or C, C′, and C″ all three are zero? Why? What will be the effect of making D, D′, and D″ zero, and $A = A' = A''$, $B = B' = B''$, and $C = C' = C''$?

8. Solve the equations,

$$x + y = a,$$
$$x + z = b.$$
$$y + z = c.$$

Ans. $y = \frac{a + c - b}{2}$, $x = \frac{a + b - c}{2}$, $z = \frac{b + c - a}{2}$.

What hypothesis will reduce these values to zero? What will make the first two equal? What all three equal?

9. Solve the three equations,

$$x + y + 2z - t = 4,$$
$$x - y - z = 2,$$
$$2x + 2y + z = a,$$

Values indeterminate. Why?

10. Solve the three equations,

$$x + y = a,$$
$$\frac{x}{a} - \frac{y}{b} = a,$$
$$bx + cy = 0.$$

Values found from first and second different from those found from second and third.

11. Solve the equations,

$$2y - x - 3(a - b) + \frac{z}{c} = \frac{a + b}{c},$$
$$y + x + z - 4(a + b) = -3(a + b),$$
$$y - x - 2(a - b) + a + z = 2a + b.$$

Ans. $y = a - b$, $x = b - a$, and $z = a + b$.

12. Solve the equations,

$$ay + bx = c,$$
$$y + x = b,$$
$$y + x - z = a.$$

Ans. $y = \frac{c - b^2}{a - b}$, $x = \frac{ab - c}{a - b}$, and $z = b - a$.

What will be the effect of making $b = a$?

13. Combine $\frac{y + x}{3} + \frac{y + z}{4} + \frac{y + t}{5} = 3.$

$$\frac{x + z}{5} + \frac{x + t}{6} + \frac{z + t}{7} = 3.$$
$$x - y + t - z = 2,$$
$$x + y + 1 + t - z = 5.$$

Ans. $y = 1$, $x = 2$, $z = 3$, and $t = 4$.

14. Combine $xy = 90,$

$$x + z = 29,$$
$$x - \frac{1}{y} = 8\tfrac{9}{10}.$$

Ans. $x = 9$, $y = 10$, and $z = 20$.

15. Combine

$$\frac{x}{3} + \frac{y}{2} + \frac{z}{4} + \frac{t}{6} = 36.$$
$$\frac{x}{3} + y + \frac{t}{6} = 36.$$
$$\frac{y}{2} + \frac{z}{2} + \frac{t}{10} = 36.$$
$$\frac{+ 6z}{40} + \frac{t}{2} = 36.$$

Ans. $x = 18$, $y = 20$, $z = 40$, and $t = 60$.

16. Combine

$$xy = 90,$$
$$x + z = 29.$$
$$x - \frac{1}{y} = 0.$$

Equations absurd. Why?

17. Combine

$$x = 2z + 4,$$
$$y = 3z - 6,$$
$$x = 2z - 2.$$

Ans. $x = \infty$, $y = \infty$, and $z = \infty$.

18. Combine

$$x + y + z + t = 4,$$
$$x + y - z - t = 0,$$
$$z + t - x - y = 0,$$
$$y + z - x - t = 0,$$
$$y - z + x - t + w = 1.$$

Ans. $x = 1$, $y = 1$, $z = 1$, $t = 1$, and $w = 1$.

19. Combine

$$x + y + z = 4,$$
$$2x + 2y + 2z = 5$$
$$y - z = 0.$$

Ans. $x = \infty$, $y = \infty$, and $z = \infty$.

Equations evidently not simultaneous. The combination, then, is absurd, and the result shows the absurdity.

20. Combine

$$x + y + z = 3,$$
$$x - y - z = 1.$$
$$3x + 3y + 3z = 18.$$

Ans. $x = \infty$, $y = \infty$, and $z = \infty$.

The first and third equations plainly conflict Hence, the equations are not simultaneous.

The first and second combined give $x=2$; and this value substituted in the second equation, gives $y+z=1$. The values of x and $y+z$, substituted in the third equation, give $9=18$.

But we may have the absurdity shown by its appropriate symbol by combining the first and third equations, and retaining the trace of one of the unknown quantities. Thus, we may have $0x=3$, or $x=\infty$. And so, likewise, y and z may be shown to be infinite.

21. Combine the equations,

$$-x+y+z+t-b+a=(a-b)(c+d+e),$$
$$x-y-z+t=a-b+(b-a)(c+d-e),$$
$$x+\frac{y}{c}+\frac{z}{d}+\frac{t}{e}=4(a-b),$$
$$\frac{x}{a-b}+\frac{y}{c(a-b)}+\frac{z}{d(a-b)}+\frac{t}{e(a-b)}-x=4+b-a.$$

Ans. $x=a-b$, $y=(a-b)c$, $z=(a-b)d$, $t=(a-b)e$.

When will all the values be zero? When three? When the last, only?

22. Combine the equations,

$$2y-3z=1,$$
$$2y+3z=7,$$
$$y+z=3,$$
$$y-z=1.$$

Ans. $y=2$, and $z=1$.

How does it happen that true solutions are found for four equations, involving but two unknown quantities?

23. Combine

$$x+y+z+t+u=200,$$
$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}+\frac{t}{5}+\frac{u}{6}=50,$$
$$x+y-z=10,$$
$$x+z-t=10,$$
$$x+t-u=10.$$

Ans. $x=20$, $y=30$, $z=40$, $t=50$, and $u=60$.

24. Combine the equations,

$$x+y+z+t+u=200,$$
$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}+\frac{t}{5}+\frac{u}{6}=50,$$
$$x+y-z=10,$$
$$x+z-t=10,$$
$$x+t-u=10,$$

(A) $y + z - u = 10$,
(B) $y + t - u = 20$.

Ans. Same as for the last equation.

In the last two examples the new equations, marked (A) and (B), were not incompatible with the others, and, therefore, the solutions found, from taking as many equations as unknown quantities, satisfied the additional equations. When, however, the number of equations exceed the number of unknown quantities, and the surplus equations are not satisfied by the same values as the other equations, the solutions will be contradictory.

General Remarks.

222. If we have m equations involving m unknown quantities, we can find the value of one of these unknown quantities in terms of the others, and this value substituted in all the other equations will give us $m - 1$, new equations, involving $m - 1$ unknown quantities. We can now take one of these $m - 1$ equations, and find the value of a second unknown quantity in terms of the remaining $m - 2$ unknown quantities, and thus get $m - 2$ new equations, involving $m - 2$ unknown quantities. And, by continuing this process, it is plain that, after $m - 1$ eliminations, we would get a single equation, involving a single unknown quantity, and, therefore, would be able to find the value of that unknown quantity. But, if we have m equations, involving $m + b$ unknown quantities, we can only make $m - 1$ substitutions, and then we will have a single equation, involving $b + 1$ unknown quantities. If $b = 1$, the final equation will contain two unknown quantities, and be of the form $x + y = 10$, an indeterminate equation, in which the value of x can only be found by attributing arbitrary values to y. If $b = 2$, the final equation will contain three unknown quantities, and be of the form of $x + y + z = 10$; in which x can only be determined by giving arbitrary values both to y and z. It is plain, then, that when the number of unknown quantities exceed the number of equations, the elimination will lead to indetermination.

223. If, on the contrary, we have $m + b$ equations, involving m unknown quantities, m of these equations are sufficient to determine the m unknown quantities. The remaining b equations might, or might not, be satisfied by the values found from the m equations. If the b equations are satisfied, they are not independent equations; if they are not satisfied, they are contradictory equations. Thus, $x = 2$, and

$2x = 4$ are satisfied by the same value of x; but the second equation is not an independent equation, since it differs only in form from the first. $x = 2$, and $x = 3$ are contradictory equations. We conclude, then, in general, that the number of independent equations must be precisely equal to the number of unknown quantities.

224. An artifice sometimes enables us to eliminate between a number of equations more readily than by the usual direct process. As an illustration take the equations,

$$\begin{aligned} x + y &= 5, \\ x + z &= 6, \\ z + y &= 7. \end{aligned}$$

Let $s = x + y + z$. The three equations then become

$$\left.\begin{aligned} s - z &= 5 \\ s - y &= 6 \\ s - x &= 7 \end{aligned}\right| \text{ A}$$

Adding, we get $3s - (x + y + z) = 18$. But, since $x + y + z = s$, we have $2s = 18$, or $s = 9$. This, substituted in equations marked A, gives $z = 4$, $y = 3$, and $x = 2$.

Take, as a second illustration,

$$\begin{aligned} xy + xz &= 27, \\ yz + y + z &= 29, \\ xyz &= 60. \end{aligned}$$

By factoring and solving the first equation, we get $y + z = \frac{27}{x}$. This, substituted in the second, gives $yz + \frac{27}{x} = 29$. But from the third equation we find $yz = \frac{60}{x}$. Hence, $\frac{60}{x} + \frac{27}{x} = 29$, from which $x = 3$. Then, $y + z = \frac{27}{x} = 9$, and $xyz = 60$, or $yz = 20$. Eliminating, we get $z^2 - 9z + 20 = 0$, or $z^2 - 10z + 25 + z - 5 = 0$, or $(z - 5)^2 + z - 5 = 0$, or $(z - 5)(z - 5 + 1) = 0$. Dividing out the second factor, we have left $z - 5 = 0$, or $z = 5$, from which $y = 4$.

Take another illustration,

$$\begin{aligned} xy + xz &= 12 \quad (1), \\ yz + z^2 &= 108 \quad (2), \\ z - x &= 8 \qquad (3). \end{aligned}$$

Subtracting (1) from (2), we get $y(z-x)+z(z-x)=96$, or $(z-x)(y+z)=96$ (4). Dividing (4) by (3), member by member, we get $y+z=12$ (5). Equation (2) may be put under the form of $z(y+z)=108$. Dividing this, member by member, by (5), we will have $z=9$. Equation (1), put under the form of $x(y+z)=12$, divided by (5), gives $x=1$. The value of z substituted in (5), gives $y=3$. Hence, $x=1$, $y=3$, and $z=9$.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

225. Many of the problems already given (Arts. 176 and 206) could have been solved as readily, or more readily, with two unknown quantities; and there are many problems which can only be solved by the use of two or more unknown quantities.

We will solve three of the problems already given, to show the manner of using two unknown quantities.

1. The sum of two numbers is a, and their difference b, what are the numbers?

Let $x=$ greater number, and $y=$ smaller. Then, by the condition of the problem, $x+y=a$, and $x-y=b$. Eliminating y by adding the two equations member by member, we get $x=\frac{a}{2}+\frac{b}{2}$. Eliminating x by subtracting the equations member by member, we have $y=\frac{a}{2}-\frac{b}{2}$. So, we see that when we know the sum and difference of two quantities, we get the greater by adding the half difference to the half sum, and the less by subtracting the half difference from the half sum.

Thus, the sum of two numbers is 20, and their difference 10; the greater is, by the formula, 15, and the smaller 5.

2. A fox is 125 of his own leaps ahead of a greyhound, and makes 6 leaps to the greyhound's 5; but two leaps of the greyhound are equivalent to 3 leaps of the fox. How many leaps will the fox take before he is overtaken by the greyhound?

Let $x=$ distance passed over by the fox, counted in terms of his own leaps. Let $y=$ distance passed by the greyhound. Then these distances would be proportional to the relative number of leaps taken by the fox and greyhound, if their leaps were equal in length. And we would have $x, y :: 6 : 5$. But since each leap of the greyhound is equivalent to $\frac{3}{2}$ leaps of the fox, the 5 leaps of the greyhound are equivalent to $5 . \frac{3}{2} = \frac{15}{2}$ leaps of the fox. Hence, $x : y :: 6 : \frac{15}{2}$, and the distance, y, passed by the greyhound, will be expressed in terms of the leaps of the fox. From the proportion we get $y = \frac{5x}{4}$, and from the condition of the problem, $y - x = 125$. Combining, we get $x = 500$, and $y = 625$ leaps of the fox.

It will be readily seen that this solution has some advantages over that with one unknown quantity.

3. Two couriers start from different points on the same road, and travel in the same direction. The forward courier travels at the rate of b miles per hour, and the rear courier at the rate of a miles per hour. They were separated by a distance of m miles at starting. How long will it be until they come together?

Let $x =$ distance travelled by forward courier, $y =$ distance travelled by the rear courier. Then, since the distances travelled must be proportional to the rates of travel, we will have $x : y :: b : a$, or $by = ax$. From the conditions of the problem, we have $y - x = m$; combining, we get $x = \frac{bm}{a - b}$, and $y = \frac{am}{a - b}$. These distances, divided by the rate of travel, will, of course, give the time elapsed before their junction. Both expressions give for the time, $\frac{m}{a - b}$, as found, when one unknown quantity was used. This solution, compared with the preceding, shows how much the use of two unknown quantities has shortened the work.

4. A carpenter wishes to saw a piece of timber, 20 feet long, into two parts, so that one of them shall be but two-thirds as long as the other. Where must he place his saw?

Ans. At 12 feet, or 8 feet from the end.

5. A carpenter wishes to saw a piece of timber m feet long, into two such parts, that one shall be the $\frac{b}{a}^{\text{th}}$ part of the other. Where must he place the saw?

Ans. At $\frac{bm}{b + a}$ feet, or $\frac{am}{b + a}$ feet from the end.

What values must b and a have to make the solutions in examples 4 and 5 the same? When will the parts be equal?

6. A colonel wishes to divide his regiment of 800 men, and 10 companies, in such a manner that the two flank companies shall contain each one-third more men than each of the central companies. What must be the number of men in each flank company, and in each central company?

Ans. 100 men in each flank company, and 75 in each central company.

7. A colonel wishes to divide his regiment, composed of a men, and b companies, in such a manner that the two flank companies shall each contain $\frac{1}{c}$ more men than each central company. What must be the composition of the flank and central companies?

Ans. Central companies, each $\frac{ac}{bc+2}$; flank companies, each $\frac{a(c+1)}{bc+2}$.

What values must a, b, and c have to make this solution the same as the last? What do the expressions become when $c = 0$? What do the central companies become in that case?

8. A man becoming insolvent, leaves \$4000 to his two creditors: to one of whom he owes \$8000, and to the other \$6000. What share ought each to have out of the \$4000?

Ans. The first \$2285$\frac{5}{7}$, the second \$1714$\frac{2}{7}$.

9. A man becoming insolvent, leaves a dollars to his two creditors. To one he owes b dollars, and to the other c dollars. What share ought each to have?

Ans. $x = \frac{ab}{b+c}$, and $y = \frac{ac}{b+c}$.

When will these values becomes equal? When the first double of the second? What do they become when he leaves nothing? What, when he leaves enough to pay his creditors?

10. A planter hired a negro-man at the rate of \$100 per annum, and his clothing. At the end of 8 months the master of the slave took him home, and received \$75 in cash, and no clothing. What was the clothing valued at? *Ans.* \$12$\frac{1}{2}$.

Verify this result.

11. A planter hired a negro-man at the rate of a dollars for c months, and was also to give the negro a year's supply of clothing. At the end of b months the negro was taken away, and the planter paid m dollars in cash, and gave no clothing. What was the clothing valued at?

Ans. $y = \frac{cm - ab}{b}$.

When will $y = 0$? When will it be negative? When infinite?

The zero solution can be explained most satisfactorily by placing $cm = ab$ under the form $\frac{a}{c} = \frac{m}{b}$. Let the pupil make a problem to explain the negative solution.

12. A planter has 500 acres of cultivated land, which he wishes to plant in such a manner that he may have twice as much cotton as corn, and three times as much corn as small grain. What division of his land must he make?

Ans. 50 acres of small grain; 150 acres of corn; and 300 acres of cotton.

13. A planter has m acres of land and wishes to cultivate it all so that he may have a times as much cotton as corn, and b times as much corn as small grain. How much of each kind must he have?

Ans. $z = \frac{m}{1 + b + ab}$ acres of small grain, $y = \frac{bm}{1 + b + ab}$ acres of corn, and $x = \frac{abm}{1 + b + ab}$ acres of cotton.

What suppositions upon a, b, and m will make this solution the same as the last? What two suppositions will make the three divisions of land equal? What will be the effect of increasing b upon the values of z, y, and x? What of increasing or decreasing m? Why does m enter into the three numerators?

14. In the year 1692, the people of Massachusetts executed, imprisoned, or privately persecuted 469 persons, of both sexes, and all ages, for the alleged crime of witchcraft. Of these, twice as many were privately persecuted as were imprisoned, and $7\frac{17}{19}$ times as many more were imprisoned than were executed. Required the number of sufferers of each kind?

Ans. 19 executed, 150 imprisoned, and 300 privately persecuted.

15. A planter has $2500 to expend in the purchase of 30 head of horses and mules. He wishes his horses all to be equal in value, and his mules all to be equal in value, but each mule to be one-fourth less valuable than each horse, and the number of mules to be twice as great as the number of horses. What must be the price of each horse, and of each mule? *Ans.* Each horse $100, each mule $75.

16. A planter has a dollars to expend in the purchase of b head of horses and mules. He wishes to have c times as many mules as horses,

but each mule to be $\frac{1}{d}$ times less valuable than each horse. What must be the price of each mule, and of each horse?

Ans. Each horse $\frac{ad\,(c+1)}{b\,(d+cd-c)}$; each mule $\frac{a\,(d-1)\,(c+1)}{b\,(d+cd-c)}$.

What values must be given to a, b, c, and d to make this solution the same as the last? How can the solution be verified? What will be the effect of making $d=1$? Why? What is the effect of making $c=1$. The expressions for the entire cost of the horses and mules are $\frac{ad}{d+cd-c}$, and $\frac{ac\,(d-1)}{d+cd-c}$. What is the effect of making $c=0$ upon these values? What of making $a=0$? How must the expressions for the price of each horse and mule be written to show that the former decreases with the increase of d, and that the latter increases with the increase of d?

17. The sum of two digits is 6, and the second digit is double the first. What is the number made up of these two digits?

Ans. 24.

The digits are the individual figures making up a number. In this example 2 is the first digit, and 4 the second.

18. The sum of two digits is a, and the second digit is b times greater than the first. What are the digits, and what is the number?

Ans. $x=\frac{a}{b+1}$, $y=\frac{ab}{b+1}$; number $=\frac{10a+ab}{b+1}$.

The number 24 is made up of the digits 2 and 4, the number 42 of those digits in reverse order. To express that, 18 added to the first number would reverse the digits; we represent by x and y the digits in the first number. Then, $10x+y+18=10y+x$.

19. A number is made up of two digits, and the first is double the second. If 27 be taken from the number, the digits will be reversed. What is the number? *Ans.* 63.

$10x+y-27=10y+x$, and $x=2y$.

20. A number is made up of two digits, and the first is a times as great as the second. And if b be taken from the number, the digits will be reversed. What is the number, and what are the digits?

Ans. First digit $\frac{ab}{9\,(a-1)}$; second $\frac{b}{9\,(a-1)}$. Number, $\frac{10ab+b}{9\,(a-1)}$, or, $\frac{(10a+b)\,b}{9\,(a-1)}$

What will these results become when $a=1$? Does the equation of the problem indicate the absurdity? What will the number become when $b=a-1$? What, when $a<1$? How is the negative solution explained?

21. A number is made up of three digits, whose sum is equal to 10. The first digit is double the second, and the second triple the third. What is the number? *Ans.* 631.

22. A number is made up of three digits, whose sum is equal to a. The first is b times as great as the second, and the second c times as great as the third. What are the digits, and what is the number?

Ans. $z=\frac{a}{bc+c+1}$, $y=\frac{ac}{bc+c+1}$, $x=\frac{acb}{bc+c+1}$; and

$$N=\frac{100acb+10ac+a}{bc+c+1}.$$

What will be the effect of increasing c upon the three digits? What of increasing b? What upon the digits and number of making $b=0$?

23. A number is made up of three digits, whose sum is equal to 10. The first digit is double the second; and if 495 be taken from the number, the digits will be reversed. What is the number?

Ans. 631.

24. The sum of the three digits which make up a number is equal to m. The first digit is a times as great as the second, and if b be taken from the number the digits will be reversed. What are the digits, and what is the number?

Ans. $z=\frac{99am-b\,(a+1)}{(2a+1)\,99}$, $y=\frac{99m+b}{(2a+1)\,99}$, $x=\frac{(99m+b)\,a}{(2a+1)\,99}$,

and number $=\frac{100a\,(99m+b)+10\,(99m+b)+99am-b\,(a+1)}{(2a+1)\,99}$.

25. A gentleman in Richmond expressed a willingness to liberate his slave, valued at \$1000, upon the receipt of that sum from charitable persons. He received contributions from 24 persons; and of these there were $\frac{14}{19}$ths fewer from the North than from the South, and the average donation of the former was $\frac{4}{5}$ths smaller than that of the latter. What was the entire amount given by the latter?

Ans. \$50 by the former; \$950 by the latter.

26. If $7\frac{1}{2}$ be taken from the numerator and denominator of a certain

fraction, its value will be doubled; but, if $6\frac{2}{3}$ be taken from the numerator and denominator, its value will be trebled. What is the fraction?

Ans. $\frac{5}{6}$.

What fraction is that from which, if a be taken from both its terms, the value will be doubled; and if b be taken from both its terms, the value will be trebled?

$$Ans.\ \frac{2a-b}{4a-3b},\ \text{or}\ \frac{\frac{ab}{4a-3b}}{\frac{ab}{2a-b}}$$

The first result gives the reduced fraction, the second (which is identical with it) the fraction in which the substitution must be made. We will illustrate by a problem.

27. Find a fraction, such, that if 5 be taken from both its terms, the value of the fraction will be doubled; but, if 4 be taken from both its terms, the value will be trebled.

$$Ans.\ \frac{3}{4},\ \text{or}\ \frac{2\frac{1}{2}}{\frac{30}{9}}.$$

The subtraction of 5 from the numerator and denominator of the second fraction will give $\frac{3}{2}$, and the subtraction of 4, in like manner, will give $\frac{9}{4}$. But these results could not be obtained by operating on the first fraction. The reason of the difference is obvious.

28. If A and C can do a piece of work in 3 days, B and C together in 7 days, and A and B together in $3\frac{3}{6}$, in what time can each person do the work alone?

Ans. A in $4\frac{1}{5}$ days; B in 21 days, and C in $10\frac{1}{2}$.

29. A and C can do a piece of work in a days, B and C can do the same in b days, and A and B the same in c days. In how many days can each one, alone, do the work?

Ans. A in $\dfrac{2abc}{bc+a(b-c)}$ days, B in $\dfrac{2abc}{ac+b(a-c)}$ days, C in $\dfrac{2abc}{bc-a(b-c)}$.

What do these values become when $a=c$? What, when $b=c$? What, when $a=b=c$? Suppose $a(b-c)>bc$, will c be a co-operator, or a draw-back?

30. If A can do a piece of work in $4\frac{1}{5}$ days, B in 21 days, and C in $10\frac{1}{2}$ days, how long will it take them all, working together, to do it?

Ans. $x = 2\frac{5}{8}$ days.

Solved by a single equation.

31. If A can do a piece of work in a days, B in b days, C in c days, how long will it take them all, working together, to perform it?

Ans. $x = \frac{abc}{ab + ac + bc}$ days.

If $b = 0$, then, $x = 0$, an absurd result. But, by going back to the equation of the problem, one of the terms is infinite, and the absurdity appears under its appropriate symbol.

32. A farmer has a piece of land, worth $800, and two negroes. The first negro and land together are worth three times as much as the second negro, and the second negro and land together are worth just as much as the first negro. What is the worth of the negroes?

Ans. First, $1600; second, $800.

33. A farmer has a tract of land, worth a dollars, and two slaves. The first slave and the land together are worth b times as much as the second slave, and the second slave and land together are just equal in value to the first slave. What is the value of the slaves?

Ans. First, $\frac{a(b+1)}{b-1}$ dollars; second, $\frac{2a}{b-1}$ dollars.

What do these values become when $b = 1$?

34. A has a number of five and three-cent pieces in his pocket; B wishes to get 24 of them, and gives A one dollar. How many pieces of each kind must he get?

Ans. 14 five-cent pieces, and 10 three-cent pieces.

35. A has two kinds of pieces of money in his pocket; the first worth a cents each; and the second b cents each. B wishes to get m of them, and gives A c cents. How many pieces of each kind must he get?

Ans. $x = \frac{am - c}{a - b}$ second kind, and $y = \frac{c - bm}{a - b}$ first kind.

Suppose $a = b$, and explain the absurdity of the solution. What is the meaning of the solution when $am = c$? What, when $am < c$? What, when $bm > c$?

36. A has two kinds of money. It takes 8 pieces of the first kind,

and $33\frac{1}{3}$ pieces of the second kind to be worth a dollar. B offers him a dollar for 27 pieces. How many pieces of each kind must he get?

Ans. 2 of the first, and 25 of the second kind.

37. A has two kinds of money. It takes a pieces of the first kind, and b pieces of the second kind to make a dollar. A dollar is offered for c pieces. How many of each kind must be given?

Ans. $\frac{a(b-c)}{b-a}$ of the first kind, and $\frac{b(c-a)}{b-a}$ of the second kind.

Explain the solution when $b=a$. When $b=c$. When $a=0$, or c.

38. A certain person has a certain sum of money, which he placed out at a certain interest. A second person has a less sum by $\$1666\frac{2}{3}$, which he puts out at one per cent. more interest than the first got, and receives the same income as the first. A third person has a less capital than the first by $\$2857\frac{1}{7}$, but invests it two per cent. more advantageously, and also receives the same income. What are the three sums at interest, and what the respective rates of interest of each?

Capitals, \$10,000, $\$8333\frac{1}{3}$, and $\$7142\frac{6}{7}$.

Rates of interest, 5 per cent., 6 per cent., and 7 per cent.

39. A gentleman invests a certain capital at a certain rate of interest. A second gentleman has a less capital by a dollars, but, by investing it at one per cent. more advantageously he derives as much income as the first. A third gentleman has a less capital than the first by b dollars, but, by investing it at two per cent. more advantageously, he also receives the same income as the first. Required the three capitals, and the three rates of interest.

Ans. Capitals, $\$\frac{ab}{2a-b}$, $\$\frac{2a(b-a)}{2a-b}$, $\$\frac{b(b-a)}{2a-b}$.

Rates of interest, $\frac{2(b-a)}{2a-b}$, $\frac{b}{2a-b}$, and $\frac{2a}{2a-b}$.

Verify these results. Discuss them when $b=a$. When $b=2a$, When $b=0$. When $a=0$, &c.

One hundred parts of gunpowder are composed of the following materials in the following proportions:

	For war.	For hunting.	For mining.
Nitre	75	78	65
Charcoal	$12\frac{1}{2}$	12	15
Sulphur	$12\frac{1}{2}$	10	20
	100	100	100

40. At the beginning of the Mexican war, the proprietor of the Dupont Mills wished to work up the materials of his powder for hunting and mining, and make war powder out of it. He removed the sulphur by sublimation, and then wished to ascertain what proportion to take of the remaining charcoal and nitre in the two specimens. What proportion ought he to have taken?

Ans. The proportion of the hunting materials to that of the mining, as $\frac{125}{156}$ is to $\frac{5}{26}$; or as 125 is to 30; or as 25 to 6.

Then, calling $25x$ the amount of hunting material in the nitre, we have $25x + 6x = 75$, or $x = \frac{75}{31}$. There ought then to be $\frac{25 \times 75}{31}$ parts of the hunting material, and $\frac{6 \times 75}{31}$ parts of the mining material to give 75 parts of nitre in the war mixture. A similar relation can be obtained for the proportion of charcoal.

41. The sum of four numbers is 107. The first, increased by 8, the second, increased by 4, the third, divided by 2, and the fourth, multiplied by 4, will all give equal results. What are the numbers?

Ans. 20, 24, 56, and 7.

42. The sum of 4 numbers is a. The first, increased by b, the second, increased by c, the third, divided by d, and the fourth, multiplied by f, will all give equal results. What are the numbers?

Ans. $\frac{(a-c)f-(1+fd)b}{(2+d)f+1}$, $\frac{(c+a)f+(df+1)c-(1+fd)b}{(2+d)f+1}$, $\frac{[2fb+(a-c)f]d}{(2+d)f+1}$, and $\frac{2b+a-c}{(2+d)f+1}$.

Verify these results by addition. Show that their sum is equal to a. Verify them by adding b to the first, c to the second, multiplying the fourth by f, and dividing the third by d. What single supposition will make the first and second equal to each other? What single supposition will make the third and fourth equal? What will make the first part zero? What effect will this hypothesis have upon the other parts?

43. Four persons owe a certain sum of money: of which the first is to pay one-third, the second one-fourth, the third one-fifth, and the fourth one-sixth. After paying a portion of the money, there is still a deficiency of \$36. What portion of it has each to pay?

Ans. The first, \$$12\frac{108}{171}$; the second, \$$9\frac{81}{171}$; the third, \$$7\frac{99}{171}$; the fourth, \$$6\frac{54}{171}$.

Let $60x$ = the proportion of the first.

44. Four persons owe a debt of a dollars: of which the first is to pay the $\frac{1}{b}$-th part; the second, the $\frac{1}{c}$-th part; the third, the $\frac{1}{d}$-th part; and the fourth, the $\frac{1}{e}$-th part. What has each to pay?

Ans. The first, $\frac{acde}{ed(c+b)+bc(e+d)}$; the second, $\frac{abde}{ed(c+b)+bc(e+d)}$; the third, $\frac{abce}{ed(c+b)+bc(e+d)}$; the fourth, $\frac{abcd}{ed(c+b)+bc(e+d)}$.

What hypothesis will make the first two results equal? What the second two? What all four? What will be the effect of making either b, c, d, or e equal to zero. How is this explained?

45. The denominator of one fraction is 4, and of a second fraction, 8; and the numerator of the second fraction is 4 times as great as the numerator of the first. The two fractions and their greatest common divisor, added together, are equal to 3. What are the numbers?

Ans. $\frac{3}{4}$, $\frac{12}{8}$, and $\frac{6}{8}$.

46. The denominator of one fraction is b, and that of a second fraction is c; the numerator of the second fraction is m times greater than that of the first, and the sum of the two fractions is equal to the least common multiple of their denominators. What are the fractions?

Ans. $\frac{\frac{b^2c^2}{c+mb}}{b}$, and $\frac{\frac{mb^2c^2}{c+mb}}{c}$.

47. A 1000 cubic inches of bronze were found to weigh 5100 ounces. A cubic inch of copper weighs $5\frac{1}{4}$ ounces, and a cubic inch of tin weighs $4\frac{1}{4}$ ounces. What was the proportion of copper and tin in the composition of bronze?

Ans. The copper to the tin as 85 to 15.

48. Some inspectors of cannon weighed m cubic inches of bronze, and found the weight to be w ounces. A cubic inch of copper weighs b ounces, and a cubic inch of tin weighs c ounces. How much copper, and how much tin was in the composition?

Ans. $\frac{w - mc}{b - c}$ ounces of copper, and $\frac{mb - w}{b - c}$ ounces of tin.

What will these values become when $b = c$? Going back to the equation of the problem, what will $b = c$ show in regard to w and mb? Suppose $c = 0$, what will the results show? Suppose $w = 0$, what will both solutions become?

VANISHING FRACTIONS.

The symbol $\frac{0}{0}$ has been interpreted to signify indetermination, and this is the true interpretation for solutions of equations of the first degree, when the symbol proceeds from two suppositions, made either upon the values found, or upon the equation of the problem. But the symbol may arise from a single hypothesis, and then it always indicates, not indetermination, but the existence of a common factor.

Take the expression, $\frac{x^2 - y^2}{x - y}$, which becomes $\frac{0}{0}$ when$=x—y$. But, by factoring the numerator, we have $\frac{x^2 - y^2}{x - y} = \frac{(x + y)(x - y)}{x - y} = x + y = 2y$, when $x = y$. The true value of $\frac{0}{0}$ in the present instance is $2y$, as shown by removing the common factor, $x - y$. Again, take the expression $\frac{x - y}{x^2 - y^2} = \frac{0}{0}$ when $x = y$. But $\frac{x - y}{x^2 - y^2} = \frac{x - y}{(x - y)(x + y)} = \frac{1}{x + y} = \frac{1}{2y}$ when $x = y$. And the true value of the vanishing fraction is again shown to be finite. But, take $\frac{x - y}{(x - y)^2} = \frac{0}{0}$ when $x = y$. Factoring, we get $\frac{x - y}{(x - y)(x - y)} = \frac{1}{x - y} = \frac{1}{0}$, or ∞ when $x = y$. And the true value of the vanishing fraction in the present instance is infinity. Take again, $\frac{(x - y)^2}{x - y} = \frac{0}{0}$ when $x = y$. Factoring, we get $\frac{(x - y)(x - y)}{x - y} = x - y = 0$ when $x = y$. So, we see, that when there is a common factor existing between the numerator and denominator of a fraction, which factor has become zero, the fraction may have either of three values, finity, infinity, or zero. To show it more generally, take the expression $\frac{P(x - a)^m}{Q(x - a)^n} = \frac{0}{0}$ when $x = a$. There may be three cases, m may be $= n$, $< n$, or $> n$.

In the first case, when $m = n$, the fraction becomes $\frac{P(x-a)^m}{Q(x-a)^m} = \frac{P}{Q}$, a finite quantity. In the second case, divided by $(x-a)^m$, the fraction becomes $\frac{P}{Q(x-a)^{n-m}}$, which is infinite, when $x = a$. In the third case, when $m > n$, dividing by $(x-a)^n$, the fraction becomes $\frac{P(x-a)^{m-n}}{Q} = 0$ when $x = a$. And as we have taken a general expression, we conclude, in general, that a vanishing fraction is one which assumes the form of $\frac{0}{0}$, in consequence of the existence of a common factor, which has become zero by a particular hypothesis, and that the true value of the fraction may be either finite, infinite, or zero.

227. When the common factor is apparent, we have only to strike it out before making our hypothesis, and we get at once the true value of the fraction. But there are many expressions, in which it is difficult to detect the common factor, and it then becomes necessary to know a process by which the common factor may be discovered. We will illustrate the process by a simple example, in which the common factor is apparent. Take the expression, $\frac{x^2 - a^2}{x - a} = \frac{0}{0}$ when $x = a$. Here, the assumed value of x is a. If, however, we make $x = a + h$, and substitute this value for x in both terms of the fraction, reduce the result to its lowest form, and then make $h = 0$, it is evident that we will have done nothing more than attribute to x the value a. Making the substitution, we get $\frac{a^2 + 2ah + h^2 - a^2}{a + h - a} = \frac{2ah + h^2}{h} = 2a + h = 2a$ when $h = 0$. The true value of the fraction $\frac{x^2 - a^2}{x - a}$ is then $2a$, when $x = a$.

As the same process is plainly applicable to all fractions in which the terms are affected with *numerical* exponents, we derive for such fractions the general

RULE.

Attribute to that term upon which the hypothesis is made the value which reduces the fraction to the form of $\frac{0}{0}$ *plus an increment* h, *reduce the result to its lowest form and then make* h = 0.

In the above example, x is the term upon which the hypothesis is made, and a the value which reduces the fraction to the form of $\frac{0}{0}$ Hence, by the rule, $x = a + h$.

228. — EXAMPLES.

1. Find the value of $\frac{x^3 - 3ax^2 + 3ax^2 - a^3}{x - a} = \frac{0}{0}$ when $x = a$.

Ans. $3a^2$.

2. Find the value of $\frac{x^2 - 3x + 2}{x^2 - x - ax + a} = \frac{0}{0}$ when $x = 1$.

Ans. $\frac{1}{a - 1}$.

3. Find the value of $\frac{x^2 + bx + x + b}{x^2 - ax + x - a} = \frac{0}{0}$ when $x = -1$.

Ans. $\frac{1 - b}{1 + a}$.

4. Find the value of $\frac{x^2 + nx^2 - mx - nm^2}{x^2 + ax - mx - am} = \frac{0}{0}$ when $x = m$.

Ans. $\frac{m\,(1 + 2n)}{a + m}$.

5. Find the value of $\frac{x^3 - 4x^2 - xy^2 + 4y^2}{4x^2 - 4yx} = \frac{0}{0}$ when $x = y$.

Ans. $\frac{(y - 4)}{2}$.

6. Find the value of $\frac{x^2 + bx - x - b}{x^3 - x^2 - x + 1} = \frac{0}{0}$ when $x = 1$.

Ans. ∞.

7. Find the value of the fraction $\frac{x^3 - bx^2 - a^2x + a^2b}{x^2 - bx - ax + ab} = \frac{0}{0}$ when $x = b$.

Ans. $x = b + a$.

8. Find the value of the fraction $\frac{x^3 - bx^2 - a^2x + a^2b}{x^2 - bx - ax + ab} = \frac{0}{0}$ when $x = a$.

Ans. $2a$.

The two last results arise from the fact of the given fraction being a double vanishing fraction. It can be put under the form of $\frac{(x^2 - a^2)(x - b)}{(x - b)(x - a)}$, which is a vanishing fraction, when either $x = b$, or $x = a$.

229. It is obvious, in all these examples, that x, minus the value of x, which makes the fraction assume the form of $\frac{0}{0}$, is the common factor. If we, then, divide both terms of the fraction by this common factor, and then attribute the appropriate value to x, we will have the true value of the vanishing fraction. But there are many expressions for which this rule fails, as will be seen more fully hereafter.

Since it is often difficult, if not impossible, without the aid of the differential calculus, to ascertain the existence of a common factor, if there be one, it becomes important to have a simple test by which we can tell whether $\frac{0}{0}$ indicates indetermination or a vanishing fraction. Take the fraction $\frac{\text{P}(x - a)^m}{\text{Q}(x - a)^n}$, which becomes $\frac{0}{0}$ by the single hypothesis $x = a$. It is evident that the expression is a vanishing fraction, and that the common factor is some power of $x - a$. But take the fraction $\frac{\text{P}(x - a)^m}{\text{Q}(x - b)^n}$, which becomes $\frac{0}{0}$ by the double hypothesis $x = a$, and $x = b$. It is plain that the expression cannot be a single vanishing fraction like those exhibited in the first six examples; and if it is a double vanishing fraction of the form exhibited in Examples 7 and 8, it cannot be a true solution to a problem of the first degree, since x cannot have two values. We then conclude, that when $\frac{0}{0}$ arises from a single hypothesis upon a solution of an equation of the first degree, it indicates the existence of a common factor. But, if it arises from two suppositions, it indicates indetermination. In conformity to this rule, we have interpreted $\frac{0}{0}$, in the problem of the couriers, to indicate indetermination, because the symbol proceeded from the double hypothesis, $m = 0$, and $a = b$.

FORMATION OF THE POWERS AND EXTRACTION OF ROOTS.

230. THE power of a quantity is the result obtained by multiplying it by itself any number of times.

Any quantity is the first power of itself.

If a quantity be multiplied by itself once, or enters twice as a factor in the result, the result is called the second power of the quantity. Thus, $2 \cdot 2 = 2^2$, or 4, and $a \cdot a = a^2$, are the second powers of 2 and a.

If a quantity be multiplied by itself twice, three times, four times, &c., or enters into the result as a factor three times, four times, &c., the result is called the third power, the fourth power, &c., of the quantity. In general, the number of multiplications of the quantity by itself is one less than the quantity which designates the power, and the number of times that the quantity enters as a factor in the result, is precisely equal to that quantity.

The quantity which designates the power is called the exponent of the power, and is written a little above and to the right of the given quantity.

Thus, $2^2 = 4$ is the second power or square of 2.
$2^3 = 8$ is the third power or cube of 2.
$2^4 = 16$ is the fourth power of 2.
a^y is the y power of a,

and indicates that a has been multiplied by itself $y - 1$ times, or that a enters as a factor y times in the expression a^y.

When no exponent is written, the first power is always understood. Thus, $2 = 2^1$, and $a + b = (a + b)^1$.

The quantity to be raised to a power may be expressed numerically, or by letters, and may be entire or fractional, positive or negative. And, since the power in every case is a product, we may define the formation of a power, to consist in finding the product arising from multiplying the quantity, by itself, a number of times one less then that indicated by the exponent of the power.

The power differs from an ordinary product, then, in this essential particular, all the factors of the power are equal.

231. *The root* of a quantity is that quantity which, multiplied by itself a certain number of times, will produce the given quantity.

When a quantity, multiplied by itself once, or taken as a factor twice, gives the given quantity; it is called the square root of the given quantity. Thus, 2 is the square root of 4, because $2 \cdot 2 = 4$; and a is the square root of a^2, because $a \cdot a = a^2$.

Raising quantities to powers is called *Involution*.

Extracting the roots of quantities is called *Evolution*.

Involution deals in equal factors. Evolution finds one of those equal factors.

232. Involution is a simple process. Evolution is more difficult, and requires particular explanation. We will begin with the simplest form of evolution, the extraction of the square root of whole numbers, which is nothing more than evolving one of the equal factors out of the product of two equal factors.

It is evident that evolution is the reverse of involution, and that we cannot extract any root without knowing how the powers of that root are formed. To demonstrate the rule, then, for the extraction of the square root of whole numbers, we must first examine and see how the square power of whole numbers is formed.

233. The first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

And their squares are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Reciprocally, the numbers of the first line are the square roots of the corresponding numbers in the second line. We see, also, that the square of any number below 10 is expressed by not more than two figures. That is, the square of units cannot give a higher denomination than tens. So, likewise, it may be shown that the square of tens cannot give a higher denomination than thousands, since the square of 99 is 9801.

The numbers, 1, 4, 9, 16, &c., and all the other numbers produced by the multiplication of a number by itself, are called *perfect squares*.

We see that there are but nine perfect squares between 1 and 100. The square roots of all numbers lying between 1 and 100 will be found between the consecutive roots of two perfect squares. Thus, the square root of 20 lies between the consecutive roots 4 and 5, being greater than the former, and less than the latter. The square root of 26 lies between the consecutive roots 5 and 6.

The square root of all numbers below 10,000 may be regarded as made up of tens and units. Thus, 99, the square root of 9801, is made up of 9 tens and 9 units. The number 32 is made up of 3 tens and 2 units. We have seen that the square of a number containing two figures could not give a higher denomination than thousands; conversely, the square root of thousands cannot give a number containing more than two figures; that is, a number containing tens and units.

234. If, then, we have to extract the square root of a number containing more than three figures, and less than five, we know that its root must contain two figures, and, therefore, be made up of tens and units. Before we can deduce a rule for the extraction of the root of thousands, we must know how thousands are derived from squaring tens and units.

Let a represent the tens, and b the units, which enter into the square root of thousands. Then, $(a+b)^2 = a^2 + 2ab + b^2$. Hence, thousands are made up of the square of the tens that enter into its root, plus the square of the units, plus the double product of the tens by the units.

Let us square the number 34, made up of 3 tens and 4 units, $= 30 + 4$. Then, 30 corresponds to a in the formula, and b corresponds to 4.

$$\begin{array}{lrcr}
\text{Hence,} & a^2 = (30^2) & = & 900, \\
& 2ab = 2\,.\,30\,.\,4 & = & 240, \\
& b^2 = (4)^2 & = & 16, \\
\hline
\text{Then, } (34)^2 = a^2 + 2ab + b^2 & & = & 1156.
\end{array}$$

235. If we were required to extract the square root of this number, we would have to reverse the process, and first take out a^2, then, $2ab$ and b^2.

The number, 1156, belonging to the denomination of thousands, its root must contain tens and units, and we must first get out the tens by extracting the square root of the square of the tens; that is, the $\sqrt{a^2}$. Now, the square of tens will give at least three figures, therefore the root of the tens cannot be sought in the two right hand figures, and we, therefore, separate them from the others by a dot, and the given number is then separated into what are

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & \\
 & 11\,.\,56 & a \quad b \\
 & 9 \quad 00 = a^2 & 30 + 4 \\
\hline
2a + b & 2 \quad 56 = 2ab + b^2 & \\
60 + 4 & 2 \quad 40 & \\
 & \quad\;\; 16 & \\
\hline
 & 0 \quad 00 \text{ Remainder.} &
\end{array}$$

called *periods* of two figures each. The square of 30 is 900, and the square of 40 is 1600. The number, 1156, falling between 900 and 1600, its root must contain 3 tens plus a certain number of units. We then extract the greatest square contained in the left hand period, 1100, and set the root, 30, on the right, after the manner of a quotient in division. We have now found a, square it, and subtract a^2 or 900, from 1156, and the remainder must be $2ab + b^2$: in the present instance equal to 256. The remainder, $2ab + b^2$, can be written $(2a + b)b$; and it is evident that, to find b accurately, we must divide by $(2a + b)$. But, as the term b within the parenthesis is unknown, we are compelled to use $2a$ as the approximate divisor. We write $2a$, or 60, on the left as a divisor, and divide 256 by it, and set the quotient b, or 4, on the right of the root found, and also on the right of the divisor. Now, multiply the two terms on the left, $2a + b$, or $60 + 4$ by b or 4, and we evidently form $2ab + b^2$, the parts entering into the remainder, 256. Subtracting the two products thus formed from 256, we find no remainder. Hence, 34 is the exact root of 1156.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & \\
 & 11\ .\ 56 & a + b \\
 & \ \ 9\ \ \ 00 = a^2 & \dot{3}\ 4 \\
\hline
2a + b & \ \ 2\ \ \ 56 = 2ab + b^2 & \\
\dot{6}\ 4 & \ \ 2\ \ \ 56 = 2ab + b^2 &
\end{array}$$

We have separated the tens from the units, to let the beginner see that he is really taking $a^2 + 2ab + b^2$ in succession, out of the given number, 1156. But we might have indicated their separation by a point above, and written $\dot{3}4$, instead of $30 + 4$, and $\dot{6}4$ instead of $60 + 4$. When the beginner is familiar with the principles, he may omit the dots, which are intended to guard him against confounding the tens with the units. He must observe, however, that this divisor being the double product of tens, is, itself, tens, and, therefore, if written out in full, would contain a cypher on its right. And since, in dividing by a number whose right hand figure is 0, we point off that figure from the right of the divisor, and also point off the right hand figure of the dividend, we must be careful to do this in dividing by the double product of the tens. In the present instance, the right hand figure, 6, of the remainder 256, must be separated from the other two figures, since, in using 6 as the divisor instead of 60, we have, in fact, pointed off 0 from the right of the divisor.

236. There is one point of considerable importance that needs some examination. In getting the second figure of the root, we used $2a$ as the approximate divisor of the remainder, $2ab + b^2 = (2a + b)\,b$,

whereas, the true divisor, to find b, is plainly $2a + b$. Our divisor being too small, the quotient, which is the second figure of the root, can never be too small, but may be too great. It is plain that, when b, in the expression $(2a + b)$, is very small in comparison with $2a$, it may be neglected. But b may be so large that the omission of it will give too great a quotient. The square of 35 is 1225; the square root of 1225 is then, of course, 35. Now, if we proceed to extract the square root of 1225, the remainder, after taking out a^2, or 900, will be found to be $325 = 2ab + b^2 = 300 + 25$. And we see that the square of the units has added 2 tens to $2ab$, the double product of the tens by the units. When, therefore, we point off 5 from the right of $32\dot{}5$, and divide 32 by 6, it is plain that the dividend, which ought to be $2ab$ (if the divisor is $2a$), is too great by 2 tens. The quotient, then, would be too great, if the 2 tens added were divisible by the divisor. Then the second figure of the root would be augmented improperly by the quotient, arising from dividing the 2 tens improperly added by 6, or $2a$. In the present case, however, if we omit the 2 tens, and divide 30 by 6, we get the same quotient, 5, as when the 2 tens are retained; their addition has not, then, affected the result.

$$\begin{array}{r|l|l} & a^2 + 2ab + b^2 & \\ & 3\dot{6}1 & a + b \\ & 1\,00 = a^2 & 1\dot{9} \\ 2a + b & 26\dot{1} = 2ab + b^2 & \\ 2\dot{9} & 18\,0 = 2ab & \\ & 8\,1 = b^2 & \\ & 00\,0 \quad \text{Remainder.} & \end{array}$$

But, square 19, and we get 361. In this case, b, in the expression $(2a + b)$, is not small in comparison with $2a$, and cannot, therefore, be neglected without affecting the result. Now, if we use $2a$ as the divisor of the remainder, $2ab + b^2$, to find b, we ought to use $2ab$ alone as the dividend; and, therefore, if we use the whole of the expression, $2ab + b^2$, our dividend is too great. Dividing 26 by 2, the quotient is 13, which is plainly absurd for the units of the root. But, we see that 180, or $2ab$, ought to have been the dividend corresponding to 20, or $2a$, as a divisor. The 26 tens is then too great by 8 tens; and, since the 8 tens added, give 4 for a quotient when divided by the 2 tens of the divisor; the second figure of the root is too great by 4, and we must write 9 as that figure, and not 13. The 8 additional tens in the 26 tens come from the square of the units, and being divisible by the divisor, $2a$, have improperly augmented the second figure of the root. Had the 8 tens not been divisible by $2a$, the second figure of the root would not have been increased at all, and the quotient of $2ab + b^2$ by $2a$ would have truly been the second figure of the root. In general, whenever the square of the units (b^2) incorporate into the remainder, $2ab + b^2$, tens which are exactly divisible by $2a$, the divisor, the second figure of the root will be too great, and must be diminished by the quotient of the incorporated tens by the $2a$ of the divisor.

237. The foregoing course of reasoning has shown that the second figure of the root may be too great, and the cause of its being too great; but, since the units of the root are unknown, the number of the tens proceeding from their square, that are incorporated with $2ab$, cannot be known. We must, then, in practice, form the product of $2a + b$ on the left by the second figure, b, of the root, and compare the result with the remainder. If the product is greater than the remainder, the second figure must be diminished until the product is equal to the remainder, or smaller than it. If the given number is an exact square, its root will be exact, and the product will be exactly equal to the remainder. When the root is not exact, the product must be made less than the remainder.

The preceding principles enable us to deduce for the extraction of the square root of whole numbers, embraced between 100 and 10,000, the following

RULE.

I. *Separate the two right hand figures from the other figures or figure of the given number, and find the greatest square contained in the left hand period, which may contain but one figure.*

II. *Set the root of this greatest square on the right, after the manner of a quotient in division. Subtract the square of the root thus found from the first period, and annex the second period to the remainder.*

III. *Double the root found and place it on the left for a divisor. Seek how often the divisor is contained in the remainder, exclusive of the right hand figure, and place the quotient on the right of the root already found, separated from it by a dot above. Place it also on the right of the divisor, separated from it in like manner.*

IV. *Multiply the divisor thus augmented by the second figure of the root, and subtract the product from the first remainder. If there is no remainder, the root is exact. If the product exceed the first remainder, the second figure of the root must be diminished until the product is equal to or smaller than the first remainder.*

EXAMPLES.

1. Extract the square root of 225. *Ans.* 15.
2. Extract the square root of 7569. *Ans.* 87.
3. Extract the square root of 2025. *Ans.* 45.
4. Extract the square root of 841. *Ans.* 29.
5. Extract the square root of 2500. *Ans.* 50.
6. Extract the square root of 7921. *Ans.* 89.
7. Extract the square root ot 9801. *Ans.* 99.
8. Extract the square root of 4096. *Ans.* 64.
9. Extract the square root of 5476. *Ans.* 74
10. Extract the square root of 7056. *Ans.* 84.

238. We have seen that the second figure of the root has frequently to be diminished. We may diminish it too much, and it becomes necessary to know when we have made the second figure too small. The test of this depends upon the principle, that the difference between two consecutive squares is equal to twice the smaller number plus unity.

Let $a =$ smaller number.
Then, $a + 1 =$ consecutive number, or the number just above a.
And $(a + 1)^2 = a^2 + 2a + 1$.
$(a)^2 = a^2$.

Their difference is $2a + 1$, as enunciated.

Now, when there is a remainder, after finding the second figure of the root, and subtracting the product of it by the quantity on the left from the first remainder, it is evident that the second remainder expresses the excess of the given number, which we may regard as $(a + 1)^2$, over the square of the two figures found. If, then, the second remainder be exactly twice the root found plus unity, it is evident that the root found is a, and that the second figure of the root can be increased by unity.

To illustrate, suppose 6 to be the square root of 49, then the remainder being equal to twice the root found plus unity, the root can be increased by unity. In general, whenever the remainder exceeds twice the root found plus unity, the root can be augmented by unity. If the remainder is exactly equal to twice the root found plus

$$\begin{array}{l|l} \multicolumn{2}{c}{(a + 1)^2} \\ 49 & a \\ 36 = a^2 & 6 \\ \hline 13 = 2a + 1 & \end{array}$$

unity, the root, increased by unity, will be the exact root of the given number.

239. This rule is of importance in finding the square root of imperfect squares. Let it be required to find the square root of 156. We find a remainder 12, and a root 12. Is 12 the greatest root contained in 156? Is the root 12 plus a remainder, or 13 plus a remainder? By the principle just demonstrated, the true root of 156 must be 12 plus a remainder, because the second remainder is not double the whole root found, plus unity.

```
   1·56 | 12
   1
22 | 56
     44
     12
```

240. This principle also enables us to pass from the square of a number to the square of a consecutive number without raising the second number to the square power. We have only to represent the smaller number by a, then the consecutive number will be $a + 1$, and its square must exceed a^2 by $2a + 1$.

Thus, $(100)^2 = a^2 = 10000$.

Then, $(101)^2 = (a + 1)^2 = \overset{a^2}{10000} + \overset{(2a + 1)}{200 + 1} = 10201$.

So, also, $(102)^2 = \overset{(a + 1)^2}{(101 + 1)^2} = \overset{a^2}{10201} + \overset{(2a + 1)}{202 + 1} = 10404$.

The following are incommensurable numbers.

EXAMPLES.

1. Find the square root of 1720. *Ans.* 41 +.
2. Find the square root of 1445. *Ans.* 38 +.
3. Find the square root of 6411. *Ans.* 80 +.
4. Find the square root of 5555. *Ans.* 74 +.
5. Find the square root of 1755. *Ans.* 41 +.
6. Find the square root of 1960. *Ans.* 44 +.
7. Find the square root of 7777. *Ans.* 88 +.
8. Find the square root of 6666. *Ans.* 81 +.

241. If we square any number, as 12 and 55, containing two figures, and made up then of tens and units, the square will contain two periods, counting from the right, and it is plain that the tens can only be sought in the periods on the left. If we square a number made up of

hundreds, tens, and units, the square will contain three periods, and the hundreds can only be found in the left hand period, and the tens only in the second period, annexed to what is left of the first periods after the square of the hundreds has been taken from it. In general, the number of periods in the given number always indicate the number of figure places in the root, and each figure of the root has its appropriate period or periods.

The principles that have been demonstrated for the extraction of the square root of numbers between 100 and 10,000 can readily be extended to any numbers whatever. Let a represent the highest denomination in the root, and s all the succeeding denominations in the root. Then the number itself will be expressed by $(a + s)^2 = a^2 + 2as + s^2$, an analogous expression to $(a + b)^2 = a^2 + 2ab + b^2$, and differing only in its more general significance. The a in one formula is not restricted to represent tens, as it is in the other, but may represent hundreds, thousands, millions, &c.; and the s of the first formula is not restricted to represent units only, but may represent tens and units, hundreds, tens, and units, &c.

Let us square 155 by means of the formula.

$$\begin{array}{llll} \text{Then,} & a = 100, \text{ and } s = 55. & & \\ & a^2 = (100)^2 & = & 10000, \\ & 2as = 200 \cdot 55 & = & 11000, \\ & s^2 = (55)^2 & = & 3025, \\ \hline \text{Hence,} & (155)^2 & = & 24025 \end{array}$$

We see from the formula that a, it matters not what may be its denomination, must first be found; and, that after its square has been subtracted from the given number, the remainder will be $2as + s^2 = (2a + s)s$.

The number, 24025, being greater than 10000, its root will be greater than 100, and, therefore, a^2 cannot be found in the two right hand periods. We seek it in the period on the left, and after placing it on the right, and subtracting its square from the given number, have 14025 for a remainder. We cut off the right hand figures, because 2, the approximate divisor, is really 200; and, after trial, we find 55 to be the right hand figures of the root.

$$\begin{array}{r|l|l} & a^2 + 2as + s^2 & \\ & 2\dot{4}0\dot{2}5 & a + s \\ & 1\,00\,00 & 1\dot{5}5 \\ \hline 2a + s & 1\,40\,25 = 2as + 0^2 & \\ 2\dot{5}5 & 1\,11\,00 = 2as & \\ & 30\,25 = s^2 & \\ & \overline{0\,00\,00} \text{ Remainder.} & \end{array}$$

The 55 is set on the right of the root already found, and also on the right of the divisor. The product of 255 by 55 is 14025, and there is, consequently, no second remainder, and the root is exact.

242. The approximate divisor is always large for numbers above 10000, and s can only be found by repeated trials. But the above process can be greatly simplified by observing that, since $2a$ enters into s, representing several denominations, it must enter into each denomination separately. Thus, in the foregoing example, $2a$, or 2, being a multiplier of 55, is a multiplier of the first 5, regarded as 5 tens, or 50, and of the second 5, regarded as units. We might, then, have found the 5 tens and the 5 units, separately taking care to write the one after the other, so as to make their denomination distinct. In the present example, $2as$ being equal to $2 . 55$, is, of course, equal to $2\ (50 + 5)$; s may be regarded as a single term, 55, to be found at once, or it may be regarded as made up of 50 and 5, separate terms, to be found separately. But, if the second and third figure of the root be found separately as independent numbers, they must be sought for in their appropriate periods. It will simplify the work when we proceed in this manner to subtract the square of a from the left hand period, and bring down each term in succession. In this case, since we make two terms of s, let $s = s' + s''$. Then the root will be $a + s'' + s''$, and the number will be $(a + s'' + s'')^2$, which, by performing the multiplication indicated, will give us $a^2 + 2as' + s'^2 + 2as'' + 2s's'' + s''^2 = a^2 + 2as' + s'^2 + 2\ (a + s')\, s'' + s''^2$. When we subtract a^2 from the left hand period, we have, in fact, subtracted 10000 from the given number.

	$(a + s' + s'')^2$	
	2 40 25	$a + s' + s''$
	1 . . $= a^2$	1 5 5
$2a + s'$	14 0 $= 2as' + s'^2 +$ &c.	
2 5	12 5 $= 2as' + s'^2$	
$2(a + s') + s''$	152 5 $= 2\ (a + s')\, s'' + s''^2$	
30 5	152 5 $= 2\ (a + s')\, s'' + s''^2$	
	0 Remainder.	

After the second period has been annexed to the remainder, the 140 truly represents 14000, and, since the zero, on the right of 140, belongs to the denomination of hundreds, it must be separated from the 14 when we come to seek for the tens of the root, because s', the tens, is sought for in $2as'$ by using $2a$ as the approximate divisor. Now, $2as'$ must be at least thousands, and, therefore, the denomination of hundreds does not contain s'. We write s', when found, on the right of the first term of the root and also on the right of the divisor. Multiplying the divisor thus augmented by s' in the root, and subtracting the product from the first remainder, we will plainly have left, after annexing the next period, $2(a + s')s'' + s''^2$. And we see that the approximate divisor to find s'' is $2(a + s')$. The whole root already found must then be doubled and used as our approximate divisor. The right hand figure of 1525 is cut off, because $2(a + s')s''$ gives at least hundreds. After s'' is found, we multiply $2(a + s')$ by it, and have no remainder; the root is then exact. We have used the broken line, in the above example, after each minuend and subtrahend to indicate that there were other numbers to follow them.

$(a + s' + s'' + s''')^2$

	2 98 59 84		$a + s' + s'' + s'''$
	1 . . .	$= a^2$	1 7 2 8
$2a + s'$	1 98	$= 2as' + s'^2 + \&c.$	
27	1 89	$= 2as' + s'^2$	
$2(a + s') + s''$	9 5 9	$= 2(a + s')s'' + s''^2 + \&c.$	
342	6 8 4	$= 2(a + s')s'' + s''^2$	
$2(a + s' + s'') + s'''$	2 7 58 4	$= 2(a + s' + s'')s''' + s'''^2 + \&c.$	
3448	2 7 58 4	$= 2(a + s' + s'')s''' + s'''^2.$	

If we have a number made up of 4 periods, as 2 98 59 84, we know that its root must contain 4 figure places, which may be represented by a, s', s'', and s'''. And the given number must then be equal to $(a + s' + s'' + s''')^2 = a^2 + 2as' + s'^2 + 2(a + s')s'' + s''^2 + 2(a + s' + s'')s''' + s'''^2$.

We see, then, that $2a$ is the approximate divisor to find s', the second figure of the root; $(2a + s')$ the approximate divisor to find s'', the third figure of the root; and $(2(a+s') + s'')$ the approximate divisor to find s''', the units of the roots. In other words, we see that the whole root found has to be doubled to find each figure of the root succeeding those already found. It is evident, too, that the right hand figure of each of the successive remainders must be cut off previous to dividing by $2a$, $2(a + s')$, &c., because in all these remainders that figure is of too low a denomination to make any part of the product of the figure of the root sought by the approximate divisor. Thus, 8, which belongs to the denomination of tens of thousands, cannot be a part of the product arising from multiplying $2a$, or 2000, by s', which is hundreds. This product must give at least hundreds of thousands.

243. It is plain that, from the manner in which the square of any number of terms is formed, the foregoing demonstrations for numbers having two, three, and four periods are general; and we, therefore, have for the extraction of the square root of any number whatever, the general

RULE.

I. *Separate the given number into periods of two figures each, beginning on the right; the left hand period may contain but one figure.*

II. *Find the greatest square contained in the left hand period, and set its root on the right, after the manner of a quotient in division. Subtract the square of the root found from the left hand period, and to the remainder annex the second period, and use the number thus found as a dividend.*

III. *Double the root found and place it on the left for a divisor. Seek how often the divisor is contained in the dividend exclusive of the right hand figure, and place the quotient on the right of the root already found, and separate the two figures by a point. Set the quotient also on the right of the divisor and separate in like manner.*

IV. *Multiply the divisor thus increased by the second figure of the root, subtract the product from the dividend, and to the remainder annex the second period of the given number. Use the remainder and annexed period as a new dividend.*

V. *Double the whole root found for a new divisor, and continue the operation as before until all the periods are brought down.*

Remarks.

244. If the last remainder is zero, the given number is a perfect square. But, if the remainder is not equal to zero, we have only found the entire part of the root sought, and the given number is incommensurable.

245. If we take 155, the square root of 24025, we observe that 15 has been derived from the two left hand periods. We might, then, after finding 5, have squared 15, and subtracted its square from 240. So, after finding the last 5 of the root, we have subtracted the square of 155 from the three given periods. In general, when we have found two figures of the root, or three figures, or four figures, &c., we may subtract their square from the two left hand periods, or the three left hand periods, &c.

GENERAL EXAMPLES.

1. Extract the square root of 16008001. *Ans.* 4001.
2. Extract the square root of 4937284. *Ans.* 2222.
3. Extract the square root of 1111088889. *Ans.* 33333.
4. Extract the square root of 197530469136. *Ans.* 444444.
5. Extract the square root of 36000024000004. *Ans.* 6000002.
6. Extract the square root of 1259631362889. *Ans.* 1122333.
7. Extract the square root of 15241383936. *Ans.* 123456.
8. Extract the square root of 16080910030201. *Ans.* 4010101.
9. Extract the square root of 123456787654321. *Ans.* 11111111.
10. Extract the square root of 12345678987654321. *Ans.* 111111111.
11. Extract the square root of 308641358025. *Ans.* 555555.

OF INCOMMENSURABLE NUMBERS.

246. An incommensurable number is one whose indicated root cannot be exactly extracted. Thus, the $\sqrt{2}$, $\sqrt{8}$, and $\sqrt{27}$ are incommensurable numbers. Such numbers are also called irrational numbers, and sometimes surds.

We have indicated (Art. 240,) that the roots of imperfect square powers were not complete by writing the sign + after the entire parts of those roots. So we may write the $\sqrt{5} = 2 +$. The number 5 lying between 4 and 9, the square roots of which are 2 and 3, its own root will be greater than 2, and less than 3. May not this root, then, be expressed by some fraction whose value is greater than 2, and less than 3, such as $\frac{5}{2}$, or $\frac{7}{3}$? May not the roots of all imperfect square powers be expressed by vulgar fractions in exact parts of unity?

To prove that this cannot be, we will demonstrate a theorem upon which depends the proof of its absurdity.

THEOREM.

247. Every number, P, which will exactly divide the product, $A \times B$, of two numbers, and which is prime with respect to one of them, will divide the other.

Let A be the number with which P is prime. Let Q be the quotient arising from dividing AB by P, then $\frac{AB}{P} = Q$. We may put this equation under the form $A \times \frac{B}{P} = Q$. Now, Q is, by hypothesis, an entire number; the second member being a whole number, the first member must also be a whole number, else we would have an irreducible fraction equal to a whole number, which is absurd. The product of A into $\frac{B}{P}$ has then to be entire; now, A itself, is entire and prime with respect to P, hence, $\frac{B}{P}$ must also be entire. For a whole number, multiplied by a fraction, can only give an entire product when the whole number is divisible by the denominator of the fraction into which it is multiplied. Thus 4, a whole number, multiplied by the fraction $\frac{3}{2}$, gives an entire product, because 4 is divisible by 2. But 5 into $\frac{3}{2}$ does not give an entire product, because 5 is not divisible by 2. Now, the whole number A is, by hypothesis, not divisible by the denominator, P, of the expression $\frac{B}{P}$; the product of A by $\frac{B}{P}$ cannot, then, possibly be equal to the whole number, Q, unless B is divisible by P.

248. We are now prepared to show that the square root of an imperfect square, such as 5, cannot be expressed by a fraction. If the root of 5 can be expressed by a fraction, let $\frac{a}{b}$ be that fraction, a being greater than b, and prime with respect to it. We assume a and b to be prime with respect to each other, because, otherwise, their quotient would be a whole number, and we know that the root of an imperfect power is found between two whole numbers. We have, then, $\sqrt{5} = \frac{a}{b}$. From which, by squaring both members, there results $5 = \frac{a^2}{b^2}$. The first member of this equation being a whole number, the second member must be a whole number also. But $\frac{a^2}{b^2}$ cannot be a whole number unless a^2 is divisible by b; for, to divide a^2 by b^2, is to divide it twice by b. Of course, then, if a^2 is not divisible by b, it cannot be divisible by b^2. But a^2 is not divisible by b, for, by the foregoing theorem, a number which exactly divides the product of two factors, and is prime with respect to one of them, must divide the

other. Now the factors of a^2 are a and a; b is prime with respect to the first factor, it must then divide the second in order to give an entire quotient. This is plainly impossible, since the second factor is the same as the first; a^2 is, then, not divisible by b; still less, then, can a^2 be divisible by b^2. The equation $5 = \frac{a^2}{b^2}$ must, therefore, be absurd; but that equation has been truly derived from the equation $\sqrt{5} = \frac{a}{b}$. A correct algebraic operation has led to an absurd result; but this can only be so when the assumption at the outset, upon which the operation is based, is absurd. The assumption, $\sqrt{5} = \frac{a}{b}$ was absurd, and it has led to an absurd result.

The foregoing reasoning has been in no way dependent upon the fact that 5 was the particular imperfect power under consideration, and is, therefore, general. We conclude, then, that the square root of no imperfect power can be expressed by an exact fraction.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

249. Since the square of a fraction is formed by squaring the numerator and denominator separately, it follows that the square root of a fraction must be taken by extracting the square root of the numerator and denominator separately.

Thus, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, since $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

So, also, $\sqrt{\frac{a^2}{b^2}} = \frac{a}{b}$, because $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$.

We may remark that all square roots can be affected with either the positive or negative sign, if these roots are regarded as algebraic quantities. Thus, the $\sqrt{\frac{4}{9}}$ may be either $+\frac{2}{3}$, or $-\frac{2}{3}$, because $-\frac{2}{3} \times -\frac{2}{3} = \frac{4}{9}$. So, also, $\sqrt{\frac{a^2}{b^2}}$ may be either $+\frac{a}{b}$, or $-\frac{a}{b}$; since $-\frac{a}{b}$, multiplied by itself, gives $\frac{a^2}{b^2}$.

EXAMPLES.

1. Extract the square root of $\frac{4}{25}$. *Ans.* $\frac{2}{5}$, or $-\frac{2}{5}$.
2. Extract the square root of $\frac{9}{16}$. *Ans.* $\frac{3}{4}$, or $-\frac{3}{4}$.
3. Extract the square root of $\frac{1}{4}$. *Ans.* $\frac{1}{2}$, or $-\frac{1}{2}$.

4. Extract the square root of $\frac{1}{9}$. *Ans.* $\frac{1}{3}$, or $-\frac{1}{3}$.
5. Extract the square root of $\frac{1}{25}$. *Ans.* $\frac{1}{5}$, or $-\frac{1}{5}$.
6. Extract the square root of $\frac{25}{4}$. *Ans.* $\frac{5}{2}$, or $-\frac{5}{2}$.
7. Extract the square root of $\frac{64}{9}$. *Ans.* $\frac{8}{3}$, or $-\frac{8}{3}$.
8. Extract the square root of $\frac{81}{16}$. *Ans.* $\frac{9}{4}$, or $-\frac{9}{4}$.
9. Extract the square root of $\frac{144}{49}$. *Ans.* $\frac{12}{7}$, or $-\frac{12}{7}$.
10. Extract the square root of $\frac{49}{4}$. *Ans.* $\frac{7}{2}$, or $-\frac{7}{2}$.

250. These examples show that the positive square roots of proper fractions are greater than the fractions themselves. Thus, $\sqrt{\frac{1}{4}} = \frac{1}{2}$. The reason of this is plain: the numerator of a proper fraction being less than the denominator, is not diminished proportionally so much as the denominator by the extraction of the square root. Let a^2 and b^2 be two unequal squares, and let $a^2 > b^2$; then, $a > b$, when a and b are both positive. Extracting the square roots of a^2 and b^2, we have a and b for the roots. The extraction of the root has then diminished a^2 a fold, and b^2 only b fold. The greater quantity has, then, been diminished the most.

251. The positive roots of improper fractions are less than the fractions themselves, because the extraction of the root diminishes their numerators more than it diminishes their denominators.

The negative roots of all fractions, whether proper or improper, are, of course, algebraically less than the fractions themselves.

252. In the foregoing examples the numerators and denominators were all perfect squares. But, if we have a fraction whose denominator is not a perfect square, we can readily find its exact root to within less than unity, divided by the denominator of the fraction. Let it be required to extract the square root of $\frac{8}{5}$. We multiply the numerator and denominator of the fraction by the denominator, which does not alter the value of the fraction, and we have $\sqrt{\frac{8}{5}} = \sqrt{\frac{8 \times 5}{5 \times 5}} = \sqrt{\frac{40}{25}} = \frac{6}{5} +$. The root of the numerator lies between 6 and 7. The root of the fraction is greater than $\frac{6}{5}$, and less than $\frac{7}{5}$; and we see that $\frac{6}{5}$ is the true value of the fraction to within less than $\frac{1}{5}$. By this, we mean that when we take $\frac{6}{5}$ as the true root of the fraction, we commit an error less than $\frac{1}{5}$. We can, of course, get a nearer approximation to the true value of the fraction by multiplying both terms of the fraction by the third, fifth, seventh, or some odd power of the denominator. This will make the denominator a perfect power, and its root can be exactly

found. Thus, $\sqrt{\frac{8}{5}} = \sqrt{\frac{8 \times 5 \times 5 \times 5}{5 \times 5 \times 5 \times 5}} = \sqrt{\frac{1000}{625}} = \frac{31}{25} +$. The true root is greater than $\frac{31}{25}$, and less than $\frac{32}{25}$. Hence, $\frac{31}{25}$ differs from the true root by a quantity less than $\frac{1}{25}$.

253. When the denominator and numerator are both imperfect powers, as in the example just given, we may make the numerator a perfect power by multiplying both terms of the fraction by the first, or some odd power, of the numerator. But, in this case, the degree of approximation is not immediately apparent. Thus, $\sqrt{\frac{8}{5}} = \sqrt{\frac{8 \times 8}{5 \times 8}} = \frac{8}{6}$ approximatively. The true root is greater than $\frac{8}{7}$, and less than $\frac{8}{6}$. The degree of approximation can only be determined by reducing these fractions to a common denominator. We have, then, $\frac{48}{42}$, and $\frac{56}{42}$, and their difference is $\frac{8}{42}$. Then, $\frac{48}{42}$ differs from the true root by a quantity less than $\frac{8}{42}$. And $\frac{56}{42}$, or $\frac{8}{6}$, also differs from the true root by a quantity less than $\frac{8}{42}$. It is plain that the degree of approximation can be more readily determined by making the denominator rational than by making the numerator rational. It is even preferable to make the denominator rational when the numerator is already so, though the process of making the denominator a perfect square make the numerator irrational. Thus, to find the approximate root of $\frac{9}{5}$, place $\sqrt{\frac{9}{5}} = \sqrt{\frac{9 \times 5}{5 \times 5}} = \sqrt{\frac{45}{25}} = \frac{6}{5} +$.

254. If the denominator is already rational, we have only to extract its root for a new denominator, and write over it the approximate root of the numerator for a new numerator.

We have, then, for finding the approximate root of any fraction, both terms of which are not rational, the following

RULE.

Make the denominator rational, if not already so, by multiplying both terms of the fraction by the first, or some odd power of the denominator, according to the degree of approximation required, so that the denominator of the given fraction shall be the square power of the denominator of the fraction that marks the degree of approximation. Then extract the root of the denominator for a new denominator, and write over it the approximate root of the numerator for a new numerator. Affect the new fraction with the double sign, to indicate that there *are two roots equal, with contrary signs. If the denominator be already rational, and a greater degree of approximation is required than that indicated by unity divided by the root of the denominator, multiply both terms of the fraction by the third, fifth, seventh power, &c., of the denominator, according to the degree of approximation required, and proceed as before.*

EXAMPLES.

1. Extract the square root of $\frac{1}{2}$ to within less than $\frac{1}{2}$ of its true value. *Ans.* $\pm \frac{1}{2}$.

2. Extract the square root of $\frac{1}{2}$ to within less than $\frac{1}{8}$ of its true value. *Ans.* $\pm \frac{5}{8}$.

3. Extract the square root of $\frac{3}{4}$ to within less than $\frac{1}{2}$ of its true value. *Ans.* $\pm \frac{1}{2}$.

4. Extract the square root of $\frac{3}{4}$ to within less than $\frac{1}{16}$ of its true value. *Ans.* $\pm \frac{13}{16}$.

5. Extract the square root of $\frac{3}{5}$ to within less than $\frac{1}{5}$ of its true value. *Ans.* $\pm \frac{4}{5}$, nearer $\frac{4}{5}$ than $\frac{3}{5}$.

6. Extract the square root of $\frac{3}{5}$ to within less than $\frac{1}{25}$ of its true value. *Ans.* $\pm \frac{19}{25}$.

7. Extract the square root of $\frac{4}{27}$ to within less than $\frac{1}{27}$ of its true value. *Ans.* $\pm \frac{10}{27}$.

8. Extract the square root of $\frac{4}{27}$ to within less than $\frac{1}{729}$ of its true value. *Ans.* $\pm \frac{280}{729}$.

In the first example, the multiplier of both terms of the fraction is 2; in the second, $(2)^5$; in the third, unity; in the fourth, $(4)^3$; in the eighth, $(27)^3$.

255. Since the denominator of the fraction may be raised to as high an even power as we please, it is evident that the degree of approximation can be made as close as we choose to the true value of the fraction.

256. We may, by a similar process, determine approximatively the roots of incommensurable numbers to within less than unity, divided by any whole number.

Let it be required to determine the square root of 2 to within less than $\frac{1}{6}$ of its true value. Then, $\sqrt{2} = \sqrt{\frac{2 \times (6)^2}{(6)^2}} = \sqrt{\frac{72}{36}} = \frac{8}{6} +$. The true value lies between $\frac{8}{6}$ and $\frac{9}{6}$, and, therefore, $\frac{8}{6}$ differs from the true value by a quantity less than $\frac{1}{6}$. We multiplied and divided the given number by the square of the denominator of the fraction that marked the degree of approximation required. This, of course, did not alter the value of the given number, it simply placed it under the form of a fraction with a rational denominator. The next step was to extract the root of the numerator to within the nearest unit, and to write the result over the exact root of the denominator.

To demonstrate a general rule applicable to any number, and true for any degree of approximation, let a be the number, and $\frac{1}{n}$ the fraction that marks the degree of approximation. Then, $\sqrt{a} = \sqrt{\frac{a \times n^2}{n^2}}$.

Let r denote the root of an^2 to within less than unity; in other words, let r denote the entire part of the root of an^2. We will then have $\sqrt{a} = \sqrt{\frac{an^2}{n^2}} > \sqrt{\frac{r^2}{n^2}}$, or $\frac{r}{n}$, and $< \sqrt{\frac{(r+1)^2}{n^2}}$, or $\frac{r+1}{n}$. The true root of a, then, lies between the numbers $\frac{r}{n}$ and $\frac{r+1}{n}$, which differ from each other by $\frac{1}{n}$. Hence, $\frac{r}{n}$ differs from the true root by a quantity less than $\frac{1}{n}$. So, also, $\frac{r+1}{n}$ differs from the true root by a quantity less than $\frac{1}{n}$. We then have a right to take either $\frac{r}{n}$, or $\frac{r+1}{n}$, for the approximate root. That one is taken to which the root lies nearest.

To find the approximate root of any number, a, to within less than $\frac{1}{n}$ of its true value, we have the following

RULE.

Multiply and divide the given number by the square of the denominator of the fraction that marks the degree of approximation. Extract the root of the numerator of the fraction thus formed to within the nearest unit, and set the result over the exact root of the denominator. Give the double sign to the root.

16

EXAMPLES.

1. Extract the square root of 2 to within less than $\frac{1}{5}$ of its true value. *Ans.* $\pm \frac{7}{5}$.

2. Extract the square root of 50 to within less than $\frac{1}{4}$ of its true value. *Ans.* $\pm \frac{28}{4}$.

3. Extract the square root of 50 to within less than $\frac{1}{50}$ of its true value. *Ans.* $\pm \frac{354}{50}$.

4. Extract the square root of 50 to within less than $\frac{1}{100}$ of its true value. *Ans.* $\pm \frac{707}{100}$.

It is obvious that, by increasing the denominator of the fraction that marks the degree of approximation, we may make the approximate indefinitely near to the true value of the root of the given number.

257. *Approximate roots of whole numbers expressed decimally.*

To extract the root of any whole number, a, to within any decimal of its true value, we have only to change the decimal into an equivalent vulgar fraction, $\frac{1}{10}$, $\frac{1}{100}$, or whatever it may be, and then multiply both terms of the fraction by 10^2, 100^2, &c. Whatever the decimal, which marks the degree of approximation, may be, it can be changed into a vulgar fraction, $\frac{1}{(10)^m}$; in which m is a positive whole number, greater by unity than the number of cyphers between the decimal point and first significant figure. Thus, $\cdot 1 = \frac{1}{(10)^1}$, and $m = 1$; $\cdot 01 = \frac{1}{(10)^2}$, and $m = 2$; $\cdot 001 = \frac{1}{(10)^3}$, and $m = 3$.

Hence, $$\sqrt{a} \doteq \sqrt{\frac{a\,(10)^{2m}}{10^{2\,m}}}.$$

Representing the entire part of the root of $a\,(10)^{2m}$ by r, there results $\sqrt{\frac{a\,(10)^{2m}}{(10)^{2m}}} > \frac{r}{(10)^m}$, and $< \frac{r+1}{(10)^m}$.

Hence, $\frac{r}{(10)^m}$ is the true root to within less than $\frac{1}{(10)^m}$; that is, $\frac{r}{(10)^m}$ differs from the true root by a quantity less than $\frac{1}{(10)^m}$. Now, multiplying the given quantity, a, by $(10)^{2m}$, is the same as annexing

$2m$ cyphers; for, multiplying it by $(10)^2$, annexes 2 cyphers; by $(10)^4$, annexes 4 cyphers; by $(10)^8$, 8 cyphers, &c. And dividing the approximate root found, r, by $(10)^m$ is plainly the same as cutting off from the right of the root found m places for decimals, for to divide the root by $(10)^1$, $(10)^2$, $(10)^3$, is the same as cutting off from the right one, two or three places of decimals. Hence, to approximate to the true root of any given number to within a certain number of decimals, we have this

RULE.

Annex twice as many cyphers to the given number as there are decimal places required in the root, extract the root of the number thus increased to within the nearest unit, and cut off from the right the required number of decimal places.

EXAMPLES.

1. Required $\sqrt{2}$ to within ·1. *Ans.* 1·4.
2. Required $\sqrt{2}$ to within ·01. *Ans.* 1·41.
3. Required $\sqrt{2}$ to within ·001. *Ans.* 1·414.
4. Required $\sqrt{50}$ to within ·01. *Ans.* 7·07.
5. Required $\sqrt{50}$ to within ·001. *Ans.* 7·071.
6. Required $\sqrt{50}$ to within ·0001. *Ans.* 7·0710.
7. Required $\sqrt{9000}$ to within ·1. *Ans.* 94·8.
8. Required $\sqrt{9000}$ to within ·01. *Ans.* ·94·86.
9. Required $\sqrt{9000}$ to within ·001. *Ans.* 94·869.
10. Required $\sqrt{145}$ to within ·01. *Ans.* 12·04.
11. Required $\sqrt{145}$ to within ·001. *Ans.* 12·041.
12. Required $\sqrt{1000}$ to within ·001. *Ans.* 31·622.
13. Required $\sqrt{1000}$ to within ·0001 *Ans.* 31·6227.
14. Required $\sqrt{100000}$ to within ·01. *Ans.* 316·22.
15. Required $\sqrt{100000}$ to within .001. *Ans.* 316·227.

Examples 12 and 14 show that, to pass from the root of any number to the root of a number 100 times as great, we have only to remove the decimal point one place further to the right. The converse is evidently true also.

MIXED NUMBERS.

258. The approximate root of mixed numbers can now readily be found. Suppose it be required to find the approximate root of 2·5 to within ·1. If we annexed two cyphers, as before, the result would be 2500; and then, when we shall have come to point off into periods, the whole number, 2, will be united with the decimal 5. The root found will be 5, which is plainly absurd. But, 2·5, changed into an equivalent vulgar fraction, is $\frac{25}{10}$. Hence, by the rule for vulgar fractions, $\sqrt{2\cdot5} = \sqrt{\frac{25}{10}} = \sqrt{\frac{250}{100}} = \frac{15}{10} + = 1\cdot5+$. We see, that in the present instance, we have annexed a single cypher, which made the decimal places even, and double the number of places required in the root. We next pointed off from the root the number of decimal places required. If we are required to find the approximate root of 2·5 to within $\frac{1}{180}$, $\frac{1}{100}$, or ·01. Then, $\sqrt{2\cdot5} = \sqrt{\frac{25}{10}} = \sqrt{\frac{25 \times 1000}{10 \cdot 10^3}} = \frac{158}{100} = 1\cdot58$. We have, obviously, added three cyphers to 5, and, therefore, made the number of decimal places even and equal to the number of places required in the root. In pointing off for decimals, we have only pointed off two places, the number required in the root. To demonstrate the rule in a general manner, let a be the entire part of the mixed number, and $\frac{b}{10^m}$ the decimal part. Then the given number will be $a \cdot \frac{b}{10^m} = \frac{ab}{10^m}$. Hence, $\sqrt{a \cdot \frac{b}{10^m}} = \sqrt{\frac{ab}{(10)^m}} = \sqrt{\frac{ab \times (10)^m}{(10)^{2m}}}$ $= \sqrt{\frac{a \times b\,(10)^m}{(10)^{2m}}} > \frac{r}{(10)^m}$, and $< \frac{r+1}{(10)^m}$. In which r represents the entire part of the root of $a \times b\,(10)^m$. Now, the multiplication of b by $(10)^m$ is the same as annexing m cyphers to b, and whenever m is odd, the number of decimal places will be even, and double the number required in the root. When m is even, the decimals will not be mixed with the whole numbers, as in the mixed number 3·45, and there need be no cyphers annexed.

We have supposed, in the general demonstration, that the number of decimal places required in the root was precisely equal to the number of decimal places in the mixed number. But, if this were not the case, the denominator of the equivalent vulgar fraction has only to be multiplied by such a power of 10 as will make it 10 with an exponent twice as great as the number of places required in the root. The numerator, when multiplied by this power of 10, will have its number of decimal places even, and equal to double the number of places required in the root.

RULE.

Annex cyphers until the number of decimal places in the mixed number is even, and equal to double the number of places required in the root. Extract the root of the result to within the nearest unit, and then point off from the right, for decimals, the number of decimal places required in the root.

EXAMPLES.

1. Extract the square root of 4·9 to within ·1 *Ans.* ± 2·2.

2. Extract the square root of 4·9 to within ·01. *Ans.* ± 2·21.

3. Extract the square root of 4·9 to within ·001. *Ans.* ± 2·213.

4. Extract the square root of 4·25 to within ·01. *Ans.* ± 2·06.

5. Extract the square root of 4·25 to within ·001. *Ans.* ± 2·061.

6. Extract the square root of 96·1 to within ·1. *Ans.* ± 9·8.

7. Extract the square root of 9·61 to within ·1. *Ans.* ± 3·1 exactly.

8. Extract the square root of 145·755 to within ·01. *Ans.* ± 12·07.

9. Extract the square root of 14575·5 to within ·1. *Ans.* ± 120·7.

10. Extract the square root of 101·7 to within ·01. *Ans.* ± 10·08.

11. Extract the square root of 1001·01 to within ·1. *Ans.* ± 31·6.

12. Extract the square root of 10·0101 to within ·01. *Ans.* ± 3·16.

13. Extract the square root of 1728·555 to within ·01. *Ans.* ± 41·57.

14. Extract the square root of 172855·5 to within ·1. *Ans.* ± 415·7.

15. Extract the square root of 17285550·666 to within ·01.
Ans. ± 4157·58.

16. Extract the square root of 1728555066·6 to within ·1.
Ans. ± 41575·8.

ROOTS OF NUMBERS ENTIRELY DECIMAL.

259. Let it be required to extract the square root of ·4. This decimal, changed into an equivalent vulgar fraction, is $\frac{4}{10}$. Hence, $\sqrt{\cdot 4} = \sqrt{\frac{4}{10}} = \sqrt{\frac{40}{100}} > \frac{6}{10}$, and $< \frac{7}{10}$. Hence, ·6 is the approximate root to within less than ·1.

We see that the number of decimal places was made even before the root was extracted.

If we were required to extract the square root of ·4 to within ·01, ·001, the denominator of the equivalent vulgar fraction must be made $(10)^4$, $(10)^6$.

The decimal fraction can be written $\frac{b}{10^m}$.

Hence, $\sqrt{\frac{b}{10^m}} = \sqrt{\frac{b \times (10)^m}{(10)^{2m}}} > \frac{r}{(10)^m}$, and $< \frac{r+1}{(10)^m}$.

It is plain that the multiplication of b by $(10)^m$ makes the number of decimal places even. In pointing off for decimals, as many places must be cut off from the right as there are periods.

RULE.

Annex cyphers to the given decimal until its places are even. Extract the root of the result, as in whole numbers, and cut off from the right, for decimals, as many places as there are periods in the number whose root was extracted. If it be required to extract the root to within a certain decimal, annex cyphers until the number of periods is equal to the number of places required in the root.

EXAMPLES.

1. Required $\sqrt{\cdot 25}$. *Ans.* ± ·5 exactly.
2. Required $\sqrt{\cdot 9}$ to within ·1. *Ans.* ± ·9.
3. Required $\sqrt{\cdot 09}$. *Ans.* ± ·3 exactly.
4. Required $\sqrt{\cdot 009}$ to within ·01. *Ans.* ± ·09.

5. Required $\sqrt{\cdot 0009}$. *Ans.* $\pm \cdot 03$ exactly.
6. Required $\sqrt{\cdot 725}$. *Ans.* $\pm \cdot 85$ exactly.
7. Required $\sqrt{\cdot 0725}$ to within ·01. *Ans.* $\pm \cdot 27$.
8. Required $\sqrt{\cdot 00725}$ to within ·001. *Ans.* $\pm \cdot 085$.
9. Required $\sqrt{\cdot 81}$. *Ans.* $\pm \cdot 9$ exactly.
10. Required $\sqrt{\cdot 0081}$. *Ans.* $\pm \cdot 09$ exactly.
11. Required $\sqrt{\cdot 00081}$ to within ·001. *Ans.* $\pm \cdot 028$.
12. Required $\sqrt{\cdot 000081}$. *Ans.* $\pm \cdot 009$ exactly.
13. Required $\sqrt{\cdot 0000081}$ to within ·0001. *Ans.* $\pm \cdot 0028$.
14. Required $\sqrt{\cdot 0144}$. *Ans.* $\pm \cdot 12$ exactly.
15. Required $\sqrt{\cdot 04937284}$. *Ans.* $\pm \cdot 2222$ exactly.
16. Required $\sqrt{\cdot 05555555}$ to within .0001. *Ans.* $\pm \cdot 2357$.
17. Required $\sqrt{\cdot 1111088889}$. *Ans.* $\pm \cdot 33333$ exactly.

These examples show that the roots of decimals are greater than the decimals themselves. This ought to be so, for all purely decimal numbers can be changed into proper fractions.

SQUARE ROOT OF FRACTIONS EXPRESSED DECIMALLY.

260. Vulgar fractions may be changed into decimal fractions, and then their roots may be extracted by the last rule. If the given fraction be mixed, it must be first reduced to an improper fraction and then changed into an equivalent decimal fraction.

EXAMPLES.

1. Required $\sqrt{\frac{2}{3}} = \sqrt{\cdot 6666} = +\cdot 81$. *Ans.* $\pm \cdot 81$.
2. Required $\sqrt{\frac{1}{4}} = \sqrt{\cdot 25}$. *Ans.* $\pm \cdot 5$ exactly.
3. Required $\sqrt{\frac{1}{9}} = \sqrt{\cdot 11111111}$. *Ans.* $\pm \cdot 3333$.
4. Required $\sqrt{\frac{1}{16}} = \sqrt{\cdot 0625}$. *Ans.* $\pm \cdot 25$ exactly.
5. Required $\sqrt{\frac{1}{81}} = \sqrt{\cdot 012345679012}$. *Ans.* $\pm \cdot 111111$.
6. Required $\sqrt{\frac{3}{4}}$ to within ·01. *Ans.* $\pm \cdot 86$.
7. Required $\sqrt{2\frac{3}{6}\frac{1}{2}\frac{2}{0}}$ to within ·01. *Ans.* $\pm 1\cdot 58$.
8. Required $\sqrt{2\frac{1}{4}}$. *Ans.* $\pm 1\cdot 5$ exactly.
9. Required $\sqrt{\frac{1}{11}}$ to within ·001. *Ans.* $\pm \cdot 301$.

10. Required $\sqrt{12\frac{5}{12}}$ to within ·001. *Ans.* $\pm$ 3·523.

11. Required $\sqrt{25\frac{1}{9}}$ to within ·001. *Ans.* $\pm$ 5·011.

12. Required $\sqrt{5\frac{1}{16}}$ to within ·01. *Ans.* $\pm$ 2·25 exactly.

The foregoing examples show that a vulgar fraction, which is a perfect square, may or may not have an exact root when changed into a decimal fraction.

261. — GENERAL EXAMPLES.

1. Required $\sqrt{48303584{\cdot}4856}$ to within ·001.
Ans. $\pm$ 6950·078.

2. Required $\sqrt{25012001{\cdot}44}$ to within ·1.
Ans. $\pm$ 5001·2 exactly.

3. Required $\sqrt{{\cdot}0289}$ to within ·01. *Ans.* $\pm$ ·17 exactly.

4. Required $\sqrt{{\cdot}000144}$ to within ·001. *Ans.* $\pm$ ·012 exactly.

5. Required $\sqrt{\frac{1}{17}}$ to within ·001. *Ans.* $\pm$ ·242.

6. Required $\sqrt{\frac{1}{64}}$ to within ·001. *Ans.* $\pm$ ·125 exactly.

7. Required $\sqrt{\frac{1}{256}}$ to within ·0001. *Ans.* $\pm$ ·0625 exactly.

8. Required $\sqrt{\frac{1}{1024}}$ to within ·00001.
Ans. $\pm$ ·03125 exactly.

9. Required $\sqrt{1728}$ to within ·001. *Ans.* $\pm$ 41·569.

10. Required $\sqrt{1728}$ to within $\frac{1}{144}$. *Ans.* $\pm$ $\frac{5986}{144}$.

11. Required $\sqrt{16\frac{5}{6}\frac{1}{1}\frac{4}{2}}$ to within ·001. *Ans.* $\pm$ 4·103.

12. Required $\sqrt{\frac{1}{25}}$ to within ·1. *Ans.* $\pm$ ·2 exactly.

13. Required $\sqrt{\frac{1}{729}}$ to within ·00001. *Ans.* $\pm$ ·03703.

14. Required $\sqrt{\frac{1}{6561}}$ to within ·00001. *Ans.* $\pm$ ·01234.

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

262. The *cube* or *third power* of a quantity is the product arising from taking the quantity three times as a factor. Thus, the cube of m is $m \times m \times m = m^3$. The cube root of a quantity is one of the three equal factors into which the quantity can be resolved. The process of extracting the cube root consists, then, in seeking one of the equal factors which make up the given quantity. When the quantity can be exactly resolved into its three equal factors, it is said to be a

perfect cube. But, when one of its factors can only be found approximatively, it is said to be an incommensurable quantity, or an imperfect cube. Thus, $\sqrt[3]{8}$, and $\sqrt[3]{27}$ are perfect cubes; but $\sqrt[3]{9}$, and $\sqrt[3]{26}$ are incommensurable, or imperfect cubes.

The first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

and their cubes

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Reciprocally, the numbers of the first line are the cube roots of the numbers of the second line.

We see, by inspection, that there are but nine perfect cubes among all the numbers expressed by one, two, and three figures. All other numbers, except the nine written above, expressed by one, two, or three figures will be incommensurable, and their roots will be expressed by whole numbers plus irrational parts, which can only be determined approximatively. Thus, the $\sqrt[3]{9}$ consists of the whole number 2, plus an irrational number. Because 9 lies between 8, whose cube root is 2, and 27, whose cube root is 3. By a course of reasoning similar to that already employed (Art. 248), it can be shown that the cube root of an imperfect cube, as 9, cannot be expressed by an exact vulgar fraction. For, if it can, let $\frac{a}{b}$ be that vulgar fraction; then, $\sqrt[3]{9} = \frac{a}{b}$, or $9 = \frac{a^3}{b^3}$. But, if $\frac{a}{b}$ be an irreducible fraction, from what has been shown, $\frac{a^3}{b^3}$ must be an irreducible fraction, and we then have a whole number equal to an irreducible fraction, which is absurd. And, since the generality of the reasoning has not been affected at all by the selection of the particular number, 9, we conclude that the root of an imperfect cube cannot be determined exactly.

Before demonstrating a rule by which the roots of perfect cubes can be determined, or the roots of imperfect cubes found approximatively, it will be necessary to examine the manner in which a cube, or third power, is formed. When the number contains less than four figures, its cube root, or the entire part of that root, must be found among the first nine figures. When the number contains more than three figures, its root must be made up of a certain number of tens and units. Let $a =$ tens of the root, and $b =$ the units of the root. Then, the number will be $(a + b)^3$. And, by actually multiplying $(a + b)$ the required number of times, we will get $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. *That is, a number, whose root is made up of tens and units, is equal to the cube of the tens, plus three times the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.*

The formula may be written $(a + b)^3 = a^3 + (3a^2 + 3ab + b^2)b$. And we see that the first thing to be done is to extract the cube root of a^3, then the tens (represented by a) will be known. It is obvious, too, that the true divisor of the remainder, after a^3 has been taken out, to find b, or the units, is the coefficient of b, $(3a^2 + 3ab + b^2)$; but, since b is unknown, the last two terms of this coefficient, $3ab$ and b^2, are unknown. Hence, we are compelled to use $3a^2$ as an approximate divisor of the first remainder to find b. Now, b, thus found, will, in every case, be too great, because the divisor has been too small; but it may happen that when b is small, the addition of $3ab + b^2$ to $3a^2$ would not diminish the quotient b by unity, and then there will be no error committed in assuming b to be the true quotient, provided we increase our divisor by $3ab + b^2$, and form the parts that enter into the first remainder.

We will illustrate by an example. Let it be required to extract the cube root of 1728.

We begin by separating the three right hand figures, because the cube of the tens must be, at least, thousands, and, therefore, cannot be contained in the right hand period.

			$a + b$
$3a^2$	$= 300$	$1\dot{}728$	$1\dot{}2$
$(3a+b)\,b = (30+2)2$	$= 64$	$1\,000 =$	a^3
$(3a^2 + 3ab + b^2)$	$= 364$	$728 =$	$(3a^2+3ab+b^2)b$
		$728 =$	$(3a^2+3ab+b^2)b$

We next seek the greatest cube in the left hand period, which is really 1000; the root is one ten, or 10, we set it on the right, after the manner of a quotient in division. $3a^2$, or 300, is assumed as an approximate divisor of the remainder after a^3, or 1000, has been taken from 1728. The remainder, 728, being represented by $(3a^2 + 3ab + b^2)b$, the true divisor to find b is, of course, the parenthetical coefficient of b. Having found b, or 2, by means of the approximate divisor, we set it on the right of the ten, separating it by a point to indicate that it is of a different denomination. We next add $(3a + b)b$, or 64, to $3a^2$, or 300, and we have 364 for the true divisor, *provided* that b has been found correctly. This divisor we will call the *supposed true divisor.* Finally, we multiply $(3a^2 + 3ab + b^2$, the *supposed true divisor* by b, and the product made up $(3a^2 + 3ab + b^2)b$, the parts entering into the remainder. The product thus formed being exactly equal to the remainder, proved two things: first, that if the true, instead of the approximate, divisor had been used, b would not have been diminished by unity; and, second, that the number 1728 is an exact cube, and that its root is 12. In the present example, it is easy to see why b was found correctly by using the approximate instead of the true divisor; a and b being both small, the omission of $3ab + b^2$ did not materially affect the divisor.

We have, from the foregoing, a simple test by which to ascertain whether b, found by using the approximate divisor, $3a^2$, must be diminished. Whenever the supposed true divisor will give a less quotient than b, we conclude that b was too great, and it must be diminished by 1, 2, &c., until the supposed true divisor will enter into the remainder the same number of times as the approximate. It will never happen when the unit figure of the root is small with respect to the tens, as in 71, 82, 93, &c, that the b, found by using $3a^2$ as a divisor, must be diminished by unity, or some greater number. But, when the unit figure is great with respect to the tens, as in 18, 19, &c., it may be necessary to diminish b by one or more units. The reason of this is plain. We will illustrate by an example. Required the cube root of 5832.

We see that b, found by using $3a^2$ as a divisor, is 16, and the supposed true divisor will then be 1036; but this, instead of entering 16 times into the remainder, 4832, will only enter 4 times. Besides, 1036, or the supposed $(3a^2 + 3ab + b^2)$, when multiplied by b, will give a product, 16576, greater than the remainder. The unit figure is then too great, and has to be diminished by 8; this diminution can only be determined by trial. The process ought to have been

$$
\begin{array}{lrr|l}
 & & 5832 & \\
3a^2 & = 300 & 1000 & 1' \\
(30+16)16 = & 736 & \overline{4832} & \\
 & (\overline{1036})\,16 = & \underline{16576} &
\end{array}
$$

$$
\begin{array}{lrl|l}
 & & 5832 & a + b \\
 & & 1000 & 1\dot{}8 \\
3a^2 & = 300 & \overline{4832} = (3a^2 + 3ab + b^2)b & \\
(30 + 8)\,8 = (3a + b)b & = 304 & \underline{4832} = (3a^2 + 3ab + b^2)b & \\
3a^2 + 3ab + b^2 & = \overline{604} & &
\end{array}
$$

The true divisor by trial was found to be 604. This divisor was represented by $(3a^2 + 3ab + b^2)$, and, when multiplied by b, or 8, gave $(3a^2 + 3ab + b^2)b$, the remainder.

Take, as another example, 6859.

$$\begin{array}{lrr|l}
 & & 6\,859 & a+b \\
3a^2 & = 300 & \underline{1\,000} & 1\dot{}9 \\
(3a+b)b=(30+9)9 & = \underline{351} & 5\,859 = (3a^2+3ab+b^2)b & \\
3a^2+3ab+b^2 & = 651 & \underline{5\,859} = (3a^2+3ab+b^2)b &
\end{array}$$

By trial, 9 was found to be the second figure of the root, the supposed true divisor was then determined to be 651, and this, multiplied by b or 9, gave 5859. The given number was then a perfect cube, and 19 its exact root. In this case, the supposed true divisor is actually the true divisor.

263. The process for extracting the cube root of a number below 10000 may be without any difficulty extended to all numbers whatever. Suppose the number to be 1881365963625, its root may still be regarded as made up of tens and units, the cube of the tens cannot enter into the last three figures, 625, on the right, and they may, therefore, be separated from the other figures. The greatest cube contained in 1881365963 must have more than one figure in its root, because the number is greater than 1000, which contains two figures in its root. Then the root of 1881365963 may be regarded as made up of tens and units; and, as the cube of the tens cannot give a less denomination than thousands, the tens cannot be found in the last three figures, 963, which may, therefore, be separated from the other figures. After the separation of 963 the foregoing reasoning may be repeated, and thus dividing the number into periods of three figures until we come, at length, to the place occupied by the cube of the tens of the highest order. The period on the left thus found, may contain but one figure, as in the present example, or it may contain two figures, or even three figures. We will designate the tens of the highest order by a', those of the second order by a'', those of the third by a''', &c. In like manner, we will designate the units of the highest order by b', those of the second order by b'', &c.

$$\begin{array}{r|l}
\dot{1}\,\dot{8}81\,\dot{3}65\,\dot{9}63\,\dot{6}25 & \dot{1}\,\dot{2}\,\dot{3}\,\dot{4}\,\dot{5} \\
1\,000 & \\
\hline
881=(3a'^2+3a'b'+b'^2)b'+ & \\
728=(3a'^2+3a'b'+b'^2)b' & \\
\hline
153\,365=(3a''^2+3a''b''+b''^2)b''+ & \\
132\,867=(3a''^2+3a''b''^2+b''^2)b'' & \\
\hline
20\,498\,365=(3a'''^2+3a'''b'''+b'''^2)b'''+ & \\
18\,213\,904=(3a'''^2+3a'''b'''+b'''^2)b''' & \\
\hline
2\,285\,059\,625=(3a''''^2+3a''''b''''+b''''^2)b'''' & \\
2\,285\,059\,625=(3a''''^2+3a''''b''''+b''''^2)b'''' & \\
\hline
\end{array}$$

$$\begin{array}{r}
3a'^2=300=\text{ 1st Ap. divisor.} \\
(30+2)2=\underline{64}=(3a'+b')b' \\
3a'^2+3a'b'+b'^2=364=\text{True divisor.} \\
64=(3a'+b')b' \\
\underline{4=b'^2} \\
3a''^2=43200=\text{2d Ap. divisor.} \\
(360+3)3=\underline{1089}=(3a''+b'')b'' \\
44289=\text{True divisor.} \\
1089=(3a''+b'')b'' \\
\underline{9=b'''^2} \\
3a'''^2=4538700=\text{3d Ap. divisor.} \\
(3690+4)4=\underline{14776}=(3a'''+b''')b''' \\
4553476=\text{True divisor.} \\
14776=(3a'''+b''')b''' \\
\underline{16=b'''^2} \\
(3a^{iv})^2=456826800=\text{4th Ap. divisor} \\
(37026+5)5=\underline{185125} \\
457011925=\text{True divisor.}
\end{array}$$

In this example, we begin by dividing the number off into periods of three figures each; the period on the left contains but one figure. We extract the greatest cube contained in this period, and set the root, 1, on the right. We then have found a', or the tens of the highest denomination. We then bring down the next period. Now, since but two periods are under consideration, the a', or 1, found, will be tens with respect to these two periods, and therefore, three times its square, or $3a'^2 = 300$. This is, then, the first approximate divisor. Dividing 881 by it, we get 2, or the supposed units contained in the root of the two left hand periods, we then add 64, or $(3a' + b')b'$, to the approximate divisor, and, if b' be the true units of the root, 364, or $3a'^2 + 3ab' + b'^2$, will be the true divisor. Now, since 364 enters into 881, the same number of times that 300 does, b' has been found correctly. We, therefore, multiply 364 by 2, and we form the three parts, $(3a'^2 + 3ab' + b'^2)b'$, of which the remainder, 881, is composed, except some tens and units which have been incorporated in it from the cube of the tens of a lower denomination. We subtract 728 from 881, and bring down the next period. Now, the next approximate divisor is plainly three times the square of 120, or 43200. But the shortest way of getting three times the square of 120 is to add 64, and the square of b', or 2, to the true divisor, and multiply the sum of the three by 100. The reason of this is apparent. The tens of the next denomination is found by dividing 153365 by 43200; the quotient 3 is set on the right of the 2 in the root, separated from it by a dot. We next add 1089, or $(3a'' + b'')b''$, to 43200, and we have the second true divisor. Multiplying this divisor by 3, and subtracting the product from 153365, and bringing down the next period, we have a new number, the root of whose unit is to be found. The approximate divisor to find b''' must be three times the square of 1230; and this is most readily found by adding 1089 and the square of b'', or 3, to 43200, and multiplying the sum by 100; that is, annexing two cyphers to the sum. The approximate divisor thus found, 4538700, enters 4 times in 20498365, and the true divisor, when formed, enters the same number of times; 4 is then, truly, the fourth figure of the root. Multiply 4538700, the true divisor, by the last figure of the root, subtract the product from 20498365, and bring down the next period. The next approximate divisor is three times the square of 12340; this can readily be formed like the preceding. The true divisor is formed when 5, the final unit, has been determined. The true divisor, multiplied by 5, gives a product equal to the last remainder. Hence, the given number is a perfect cube, and 12345 is its exact root.

The reason of the above process becomes very plain upon a slight examination. It was established at the outset, that the tens and units contained in the first two periods could be sought, independently of the other periods. Having found found 1 ten and 2 units in the first two periods, it is plain that this number, 12, is the tens in 123, the root found in the first three periods. So, 123 may be regarded as the tens of the first four periods. The root found in these periods is 1234, made up of 123 tens and 4 units. In like manner, 1234 may be regarded as the tens of the root of the whole five periods. In fact, the root found, 12345, is made up of 1234 tens and 5 units.

We then had tens of different denominations, 1 ten, 12 tens, 123 tens, and 1234 tens. To apply the general formula, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, for the cube of a number made up of tens and units, to determining the tens and units contained in any two consecutive periods, it became necessary to distinguish the tens by dashes to indicate the denomination to which they belonged.

We will add another example, to show more fully the application of the preceding principles. Required the cube root of 997002999.

$3a'^2 =$	24300 = App. divisor.	997˙002˙999	9˙9˙9
$(3a' + b')b' =$	2511	729	
	26811 = True divisor.	268 002	
	2511	241 299	
	81	26 703 999	
	2940300 = 2d App. divisor.	26 703 999	
$(3a'' + b'')b'' =$	26811		
	2967111 = 2d True divisor.		

In this example, the quotient of the division of 268002, by the approximate divisor, 24300, is 10; but this is manifestly absurd. By trial, 9 is found to be the second figure of the root, because the true divisor, 26811, enters 9 times, neither more nor less, in 268002.

We have for the extraction of the cube root of any number above 1000, the following

RULE.

I. *Begin on the right and divide off periods of three figures each. There will remain on the left a period of one, two, or three figures.*

II. *Extract the greatest cube contained in the left hand period, and set the root on the right, after the manner of a quotient in division. Subtract the cube of the root from the left hand period, and bring down the next period.*

III. *Take three times the square of the root found, regarded as tens, and set it on the left as an approximate divisor; see how often this enters into the first remainder. The quotient will be the second figure of the root, or something greater. Add to the approximate divisor the product of three times the first figure of the root, regarded as tens, plus the second figure of the root by the second figure of the root. The sum of this product and the approximate divisor will be the true divisor if it enter into the remainder the same number of times as the approximate divisor. Multiply the true divisor by the second figure of the root, subtract the product from the first remainder, and bring down the next period.*

IV. *Add to the first true divisor the same number that was before added to the approximate divisor, plus the square of the second figure of the root, and annex two cyphers to the sum, the result will be the second approximate divisor. The true divisor is found as before. Continue this process until all the periods are brought down; then, if the last true divisor, multiplied by the final units of the root, gives a product exactly equal to the last remainder, the given number is a perfect cube, and its exact root has been found.*

EXAMPLES.

1. Required the cube root of 9261. *Ans.* 21.
2. Required the cube root of 85184. *Ans.* 44.
3. Required the cube root of 8024024008. *Ans.* 2002.
4. Required the cube root of 1371330631. *Ans.* 1111.

5. Required the cube root of 1030607060301. *Ans.* 10101.

6. Required the cube root of 1367631. *Ans.* 111.

7. Required the cube root of 1879080904. *Ans.* 1234.

8. Required the cube root of 95306219005752. *Ans.* 45678.

9. Required the cube root of 95306219005752000. *Ans.* 456780.

10. Required the cube root of 468373331096. *Ans.* 7766.

APPROXIMATE ROOTS OF INCOMMENSURABLE NUMBERS.

264. Let a represent any incommensurable number—we have seen that its root cannot be expressed by an exact fraction; it may, however, be truly determined to within less than any fraction, $\frac{1}{n}$, whose numerator is unity. For $a = \frac{an^3}{n^3}$, and $a > \frac{r^3}{n^3}$, and $< \frac{(r+1)^3}{n^3}$; r denoting the entire part of the root of an^3. Then, since a is comprised between $\frac{r^3}{n^3}$, and $\frac{(r+1)^3}{n^3}$, its root will be greater than $\frac{r}{n}$, and less than $\frac{r+1}{n}$. Hence, $\frac{r}{n}$ differs from the true root by a quantity less than $\frac{1}{n}$. Now, as $\frac{1}{n}$ may be made indefinitely small, the approximate root may be found as near to the true root as we please; the difficulty of extracting the root increasing, however, with the increase of n.

Hence, we deduce the following

RULE.

Multiply and divide the number by the cube of the denominator of the fraction that marks the degree of approximation; extract the root of the new numerator to within the nearest unit, and divide the result by the root of the new denominator, which will be the denominator of the fraction that determines the degree of approximation.

EXAMPLES.

1. Required the cube root of 4 to within $\frac{1}{5}$. *Ans.* $\frac{8}{5}$.

Because $\sqrt[3]{4} = \sqrt[3]{\frac{4\,.\,5^3}{5^3}} = \sqrt[3]{\frac{500}{5^3}} < \frac{8}{5}$, and $> \frac{7}{5}$. The true root lies, then, between $\frac{7}{5}$ and $\frac{8}{5}$, but is nearer to $\frac{8}{5}$ than to $\frac{7}{5}$. Hence, $\frac{8}{5}$ is the true root to within less than $\frac{1}{5}$.

2. Required the cube root of 80 to within $\frac{1}{12}$. *Ans.* $\frac{51}{12}$.

Because $\sqrt[3]{80} = \sqrt[3]{\frac{80 \cdot 12^3}{12^3}} = \sqrt[3]{\frac{138240}{12^3}} > \frac{51}{12} < \frac{53}{12}$.

3. Required the cube root of 90 to within $\frac{1}{4}$. *Ans.* $\frac{18}{4}$.

4. Required the cube root of 712 to within $\frac{1}{4}$, *Ans.* $\frac{36}{4}$ nearly.

5. Required the cube root of 1820 to within $\frac{1}{5}$. *Ans.* $\frac{61}{5}$.

6. Required the cube root of 200 to within $\frac{1}{2}$. *Ans.* $\frac{11}{2}$ or $\frac{12}{2}$.

7. Required the cube root of 2397 to within $\frac{1}{4}$. *Ans.* $\frac{53}{4}$.

8. Required the cube root of 1531 to within $\frac{1}{2}$. *Ans.* $\frac{23}{2}$.

9. Required the cube root of 3575 to within $\frac{1}{5}$. *Ans.* $\frac{76}{5}$.

10. Required the cube root of 3375 to within $\frac{1}{5}$.
Ans. $\frac{75}{5}$, or 15, commensurable.

11. Required the cube root of 1062208 to within $\frac{1}{2}$. *Ans.* $\frac{204}{2}$.

CUBE ROOT OF FRACTIONS.

265. Since the cube, or third power, of a fraction is formed by raising the numerator and denominator separately to the third power, the cube root of a fraction can plainly be found by extracting the root of the numerator and denominator separately. The root of the numerator written over the root of the denominator will then be the root of the fraction.

There are three cases: 1. The numerator and denominator of the given fraction may be both perfect cubes, and then the root of the one written over the root of the other will be the required root. 2. The numerator may be an imperfect cube, and the denominator a perfect cube, then the root of the numerator extracted to within the nearest unit, written over the exact root of the denominator, will be the approximate root to within less than unity, divided by the root of the denominator. 3. Both the numerator and denominator may be imperfect cubes, or the denominator only may be an imperfect cube, then the denominator must be made a perfect cube by multiplying it by its second power. The numerator must also be multiplied by the same number, otherwise the value of the fraction will be altered. The root of the new numerator, to within the nearest unit written over the exact root of the new denominator, will be the approximate root required. If a nearer degree of approximation be required, both terms of the fraction may be multiplied by the 5th, 8th, &c. power of the denominator. The reason for making the denominator rational rather than the numerator, is the same as that given in the explanation of the principles involved in extracting the square root of fractions.

The rule for the extraction of the cube root of fractions belonging to either of the three foregoing classes, is as follows:

RULE.

Make the denominator rational, if not already so, by multiplying both terms of the fraction by the square power of the denominator. Extract the root of the new numerator to within the nearest unit, and write the root found over the root of the new denominator, which will be the same as the denominator of the given fraction. If a nearer degree of approximation be required, both terms of the fraction may be multiplied by the 5th, 8th, &c. powers of the denominator. The root of the new fraction will be the root required.

EXAMPLES.

1. Required the cube root of $\frac{8}{27}$. *Ans.* $\frac{2}{3}$.

For $(\frac{2}{3})^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$.

2. Required the cube root of $\frac{66}{27}$ to within $\frac{1}{3}$. *Ans.* $\frac{4}{3}$.

Because $(\frac{4}{3})^3 = \frac{64}{27}$, and $(\frac{5}{3})^3 = \frac{125}{27}$. Hence, the true root lies between $\frac{4}{3}$ and $\frac{5}{3}$; and $\frac{4}{3}$, therefore, differs from the true root by a quantity less than $\frac{1}{3}$.

3. Required the cube root of $\frac{2}{5}$ to within $\frac{1}{5}$. *Ans.* $\frac{4}{5}$.

Because $\sqrt[3]{\frac{2}{5}} = \sqrt[3]{\frac{2 \cdot 5^2}{5 \cdot 5^2}} = \sqrt[3]{\frac{50}{5^3}} > \frac{3}{5}$, and $< \frac{4}{5}$. The true root is nearer $\frac{4}{5}$ than $\frac{3}{5}$, and, therefore, $\frac{4}{5}$ is taken.

4. Required the cube root of $\frac{8}{5}$ to within $\frac{1}{5}$. *Ans.* $\frac{6}{5}$.

Because $\sqrt[3]{\frac{8}{5}} = \sqrt[3]{\frac{8 \cdot 5^2}{5^3}} = \sqrt[3]{\frac{200}{5^3}} > \frac{5}{5}$, and $< \frac{6}{5}$.

5. Required the cube root of $\frac{8}{5}$ to within $\frac{1}{125}$. *Ans.* $\frac{146}{125}$.

6. Required the cube root of $\frac{13}{2}$ to within $\frac{1}{8}$. *Ans.* $\frac{15}{8}$.

266. The cube root of a fraction may also be determined to within unity, divided by any power of the denominator; we have only to multiply both terms of the given fraction by such a number as will make the denominator a perfect third power of the denominator of the fraction that marks the degree of approximation, and then extract the root of the new fraction. Let it be required to extract the cube root of $\frac{13}{2}$ to within $\frac{1}{4}$, then the given fraction must have its numerator and denominator multiplied by 32, in order to make the denominator 64, the cube of 4, the denominator of the fraction that marks the degree of approximation.

Then, $\sqrt[3]{\frac{13}{2}} = \sqrt[3]{\frac{13 \times 32}{64}} = \sqrt[3]{\frac{416}{64}} < \frac{7}{4}$, and $> \frac{6}{4}$.

8. Required the cube root of $\frac{10}{3}$ to within $\frac{1}{6}$. *Ans.* $\frac{9}{6}$.

For $\sqrt[3]{\frac{10}{3}} = \sqrt[3]{\frac{10 \,.\, 72}{216}} = \sqrt[3]{\frac{720}{216}} > \frac{8}{6}$, and $< \frac{9}{6}$.

APPROXIMATE ROOT TO WITHIN A CERTAIN DECIMAL.

267. There are two cases: the given number whose root is to be found may be entire, or it may be mixed; partly entire and partly decimal.

CASE I.

Approximate root of whole numbers to within a certain decimal.

If the decimal fraction that marks the degree of approximation be changed into an equivalent vulgar fraction, the cube of its denominator will be unity, followed by three, six, nine, twelve, or some multiple of three cyphers. In other words, the cube of the denominator will contain unity, followed by as many periods of three cyphers each, counting from the right, as there are decimal places in the fraction of approximation. Thus $(\cdot1)^3 = (\frac{1}{10})^3 = \frac{1}{1000}$. $(\cdot01)^3 = (\frac{1}{100})^3 = \frac{1}{1000000}$. Then, to multiply the given number by the cube of the denominator of the decimal changed into a vulgar fraction is nothing more than annexing three, six, nine, or some multiple of three cyphers. After a new number has thus been formed, the extraction of the root is, of course, performed just as when the fraction of approximation was a vulgar fraction. Let it be required to extract the cube root of 5 to within ·01.

$$\text{Then, } \sqrt[3]{5} = \sqrt[3]{\frac{5 \,.\, (100)^3}{(100)^3}} = \sqrt[3]{\frac{5000000}{(100)^3}} > \tfrac{170}{100}, \text{ and } < \tfrac{171}{100}.$$

RULE.

Annex to the given number three times as many cyphers as there are decimal places required in the root. Extract the root of the new number thus formed to within the nearest unit, and point off from the right the required number of decimal places; which amounts to the same thing as dividing the root of the new number by the denominator of the fraction of approximation changed into an equivalent vulgar fraction.

EXAMPLES.

1. Required the cube root of 60 to within ·1 *Ans.* 3·9.

2. Required the cube root of 1775 to within ·1. *Ans.* 12·1.

3. Required the cube root of 9 to within ·01. *Ans.* 2·08.

4. Required the cube root of 9 to within ·0001. *Ans.* 2·0801 nearly.

5. Required the cube root of 1864967 to within ·1. *Ans.* 123·1.

CASE II.

268. *Approximate roots of mixed numbers to within a certain decimal.*

Let it be required to extract the cube root of 2·3 to within ·1.

$$\text{Then, } \sqrt[3]{2{\cdot}3} = \sqrt[3]{\tfrac{23}{10}} = \sqrt[3]{\frac{23 \,.\, 10^2}{10^3}} = \sqrt[3]{\frac{2300}{10^3}} > \tfrac{13}{10}, \text{ or } > 1{\cdot}3.$$

We see that when the decimal has been changed into a vulgar fraction, and the denominator made rational, the numerator, 2300, is the given number, with the point omitted, and with cyphers enough annexed to make the number of places, counting from where the point was, a multiple of three. If it had been required to extract the cube root of 2·3 to within ·01, it would have been necessary to add five cyphers to 3; the number of decimal places then would have been made a multiple of 3. Now, in pointing off for decimals, it is plain that we point off as many places for decimals as there are places required in the root. For, if one decimal place be required in the root, the denominator of the root of the equivalent vulgar fraction will be 10; if two places be required, it will be 100, &c.

RULE.

Annex cyphers to the decimal part of the mixed number until there are three places, if one place be required in the root; six places if two be required in the root, &c. Extract the root of the new number thus formed as a whole number, and point off from the right the required number of decimals.

EXAMPLES.

1. Required the cube root of 2·12 to within ·01. *Ans.* 1·28.

For, $\sqrt[3]{2\cdot12} = \sqrt[3]{\frac{212}{100}} = \sqrt[3]{\frac{212 \times 100^2}{100^3}} = \sqrt[3]{\frac{2120000}{100^3}} > \frac{128}{100}$, or 1·28.

2. Required the cube root of 4·1 to within ·1. *Ans.* 1·6.

For, $\sqrt[3]{4\cdot1} = \sqrt[3]{\frac{41}{10}} = \sqrt[3]{\frac{41 \times 10^2}{10^3}} = \sqrt[3]{\frac{4100}{10^3}} > \frac{16}{10}$, and $< \frac{17}{10}$.

3. Required the cube root of 8·88 to within ·01. *Ans.* 2·07.

4. Required the cube root of 68·64 to within ·1. *Ans.* 4·1.

5. Required the cube root of 1770·25 to within ·1. *Ans.* 12·1.

6. Required the cube root of 1150·455 to within ·1. *Ans.* 10·4 nearly.

7. Required the cube root of 5011·125 to within ·1. *Ans.* 17·1.

In the 4th and 5th examples, one cypher only had to be annexed. In the last two examples, no annexation was required. But if the root in the last two examples is to be determined within ·01, then three additional cyphers must be annexed; if within ·001, six additional cyphers, &c.

The reason for annexing cyphers until the decimal places can be separated into periods of three figures, is evident, even without changing the decimal into an equivalent vulgar fraction. For each period of three figures must give one figure in the root; if, then, the decimal places were not made multiples of three, when we come to point off from the right the decimals would be mixed with the whole number. Suppose we were required to find the root of 8·72 to within ·1. Now, 8·72 is but little greater than 8, whose root is 2. Hence, the root of 8·72 ought to be but little greater than two; but if we annexed no cypher to 72, we would have but one period, 872, and the root would be 9 approximatively.

APPROXIMATE ROOT OF DECIMAL FRACTIONS TO WITHIN A CERTAIN DECIMAL.

269. A decimal fraction changed into an equivalent vulgar fraction will have a rational denominator when its number of decimal places are multiples of three.

Thus, $\cdot 021 = \frac{21}{1000}$, $\cdot 007681 = \frac{7681}{1000000}$, $\cdot 123456789 = \frac{123456789}{1000000000}$.

Hence, if cyphers be annexed to the decimal until the number of places are made multiples of three, the new fraction, when changed into an equivalent vulgar fraction, will have a rational denominator. And, since every period of three figures gives one figure in the root, cyphers must be annexed until the number of periods of three figures is exactly equal to the number of places required in the root.

RULE.

Annex cyphers to the given decimal until it can be divided off into as many periods of three figures each as there are places required in the root. Extract the root of the new decimal thus formed to within the nearest unit, and point off from the right the required number of decimal places.

EXAMPLES.

1. Required the cube root of ·8 to within ·1. *Ans.* ·9.

For, $\sqrt[3]{\cdot 8} = \sqrt[3]{\frac{8}{10}} = \sqrt[3]{\frac{800}{10^3}} > \frac{9}{10}$, and $< \frac{10}{10}$.

2. Required the cube root of ·08 to within ·1. *Ans.* ·4.

For, $\sqrt[3]{\cdot 08} = \sqrt[3]{\frac{8}{100}} = \sqrt[3]{\frac{80}{1000}} > \frac{4}{10}$, and $< \frac{5}{10}$.

3. Required the cube root of ·008 to within ·1.

Ans. ·2, commensurable.

For, $\sqrt[3]{\cdot 008} = \sqrt[3]{\frac{8}{1000}} = \frac{2}{10} = \cdot 2$

4. Required the cube root of ·8 to within ·01. *Ans.* ·92.

5. Required the cube root of ·8 to within ·001. *Ans.* ·928.

6. Required the cube root of ·68 to within ·1. *Ans.* ·8.

7. Required the cube root of ·68 to within ·01. *Ans.* ·87.

8. Required the cube root of ·068 to within ·1. *Ans.* ·4.

9. Required the cube root of ·0999 to within ·01. *Ans.* ·46.

10. Required the cube root of ·999 to within ·01. *Ans.* ·99.

11. Required the cube root of ·0125 to within ·01. *Ans.* ·23.

12. Required the cube root of ·125. *Ans.* ·5, commensurable.

In some of the examples, it was necessary to annex one cypher, in others, two cyphers; in others, again, three cyphers. It will be seen that the root is greater than the number; this ought to be so, since the denominator of the decimal changed into a vulgar fraction is greater than the numerator.

APPROXIMATE ROOT OF VULGAR FRACTIONS TO WITHIN A CERTAIN DECIMAL.

270. Any vulgar fraction may be changed into an equivalent decimal fraction by annexing cyphers to the numerator, and dividing the new numerator thus formed by the denominator. The equivalent decimal will be mixed, or purely decimal, according as the given fraction is improper or proper. The division of the new numerator by the given denominator must be continued until the decimal part of the quotient contains as many periods of three figures as there are places required in the root.

RULE.

Change the vulgar fraction into an equivalent decimal fraction, and make the decimal part of the quotient contain as many periods of three figures as there are places required in the root. Then extract the root according to the preceding rules.

EXAMPLES.

1. Required the cube root of $\frac{1}{4}$ to within ·1. *Ans.* ·6.
2. Required the cube root of $\frac{5}{3}$ to within ·1. *Ans.* 1·1.
3. Required the cube root of $\frac{8}{5}$ to within ·01. *Ans.* 1·17 nearly.
4. Required the cube root of $\frac{5}{9}$ to within ·01. *Ans.* ·82.
5. Required the cube root of $\frac{1}{12}$ to within ·001. *Ans.* ·436.

General Remarks on the Extraction of the Cube Root.

271. In extracting the cube root, we formed the three parts of which each successive remainder was composed, regard being had to the denomination of the tens and units in those remainders. But, since any number, when cubed, gives as many periods of three figures each, counting from the right, as there are places of figures in the given number (the period on the left, however, not necessarily containing more than one or two figures), it is plain that the first two periods on the left of the given number give the first two figures on the left of the root. The first three periods on the left give, in like manner, the first three figures on the left of the root, and so on. It is evident, then, that it is not necessary to form the three parts of which the successive remainders are composed; we have only, when the first figure of the root has been found, to cube it, and subtract the cube from the left hand period. When the second figure of the root has been found by using the approximate divisor, $3a'^2$, we have only to cube the two figures found, and subtract the cube from the two left hand periods, and bring down the next period, and so on. If the cube of the first two figures of the root exceed the first two periods on the left, the second figure of the root must be diminished until the cube of the two figures is equal to, or less than the two periods on the left. The successive figures of the root can only then be ascertained by trial. We can always tell, by cubing the two, three, or four figures of the root found, when the last figure is too great; but some test is necessary to point out when we have diminished the last figure too much. That test depends upon the principle, that the difference between two consecutive numbers is equal to three times the square of the smaller number, plus three times the smaller number, plus unity.

Let a be the smaller number, then the consecutive number next above it will be $a+1$. And $(a+1)^3-(a)^3=a^3+3a^2+3a+1-a^3$ $=3a^2+3a+1$, as enunciated. If, then, after subtracting the cube of the figures found from the periods on the left, the remainder is exactly equal to three times the square of the root found, plus three times this root, plus unity, the last figure of the root can be increased exactly by unity. If the remainder is greater than this quantity, the last figure can be increased by unity, or something more than unity.

The following examples will illustrate the foregoing process. Required the cube root of 531441.

$$\begin{array}{rl|l} & 531\dot{}441 & a+b \\ 3a^2=19200 & 512 & 8\dot{}1 \\ & \overline{19\,441} & \\ & 531\dot{}441=(a+b)^3. & \end{array}$$

After finding the tens of the root, the approximate divisor entered once in the remainder, the quotient was set on the right of the root found, and the two figures of the root (81), when cubed, gave the original number. Hence, the number was a perfect cube, and 81 its exact root.

Take, as a second example, 681472.

$$\begin{array}{rl|l} & 681\dot{}472 & a+b \\ 3a^2=19200 & 512 & 8\dot{}8 \\ & \overline{169\,472} & \\ & 681\dot{}472=(a+b)^3 & \end{array}$$

The approximate divisor gives 9 for the second figure of the root; but 89, when cubed, exceeds the given number, 681472. The last figure, must, therefore, be diminished, if we make this figure, 7, and cube 87, the cube when subtracted from 681472, leaves a remainder exactly equal to three times the square of 87, plus three times 87, plus unity. The last figure, then, can be increased exactly by unity, and 88 is the true root.

272. The formula for the difference between consecutive cubes, enables us to pass from the cube of one number to the cube of the number next above it without actually cubing the higher number. Thus, since, the cube of 10 is 1000, the cube of 11 must be $1000+3\,(10)^2+3\,(10)+1=1000+300+30+1=1331$. In like manner, since $(64)^3=262144$; then $(65)^3=262144+3\,(64)^2+3\,(64)+1=262144+12288+192+1=274625$. When the numbers

under consideration are very large, the formula will save a great deal of trouble in deducing the cube of the higher number.

The second process that we have given for extracting the cube root is usually followed, and when the given number contains but two periods, is shorter than the first process; but in every other case is longer.

EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.

273. We will first show how to extract the root of monomials.

Since all rules for the extraction of roots must be founded upon the rules for the formation of powers, we must first examine, in the present case, how the power of a monomial is formed. Let P represent the numerical coefficient of any monomial, and a^m the literal part of the monomial. Then, the monomial will be Pa^m; now, to square Pa^m, is to multiply it once by itself. And $Pa^m \times Pa^m = P^2a^{2m}$. Hence, to square a monomial, we have only to square the coefficient, and to double the exponent of each literal factor. The square of $2a^2c$ is, by the rule, $4a^4c^2$. The square of $5a^{-1}c^2d^{-2}$ is, by the rule, $25a^{-2}c^4d^{-4}$. The square root of $4a^4c^2$ is then, of course, $2a^2c$; but $-2a^2c$ will, when multiplied by itself, give $4a^4c^2$. Hence, the root of $4a^4c^2$ may be either $+2a^2c$, or $-2a^2c$. So, in like manner, the root of $25a^{-2}c^4d^{-4}$ may be either $+5a^{-1}c^2d^{-2}$, or $-5a^{-1}c^2d^{-2}$.

RULE.

I. *Extract the square of the coefficient, and divide the exponent of each literal factor by* 2.

II. *Write, after the root of the coefficient, each literal factor, with its new exponent.*

III. *Affect the whole result with the double sign,* $\pm$.

EXAMPLES.

1. Extract the square root of $16a^4c^{-6}d^{-12}$. *Ans.* $\pm 4a^2c^{-3}d^{-6}$.
2. Extract the square root of $16a^{2m}b^{4n}c^{2p}$. *Ans.* $\pm 4a^mb^{2n}c^p$.
3. Extract the square root of $144a^8b^{12}c^{16}$. *Ans.* $\pm 12a^4b^6c^8$.
4. Extract the square root of $36a^{16}b^{18}c^{20}d^{22}$. *Ans.* $\pm 6a^8b^9c^{10}d^{11}$.

5. Extract the square root of $a^{4x}b^{8y}c^{-10z}$. *Ans.* $\pm\, a^{2x}b^{4y}c^{-5z}$.

6. Extract the square root of $256\,\dfrac{a^{2n}b^{4m}c^{6x}}{d^{8z}}$.
Ans. $\pm\, 16a^{n}b^{2m}c^{3x}d^{-4z}$.

7. Extract the square root of $\dfrac{1}{a^2b^4c^6}$. *Ans.* $\pm\, a^{-1}b^{-2}c^{-3}$.

8. Extract the square root of $\dfrac{25}{x^2y^2z^2}$. *Ans.* $\pm\, 5x^{-1}y^{-1}z^{-1}$.

9. Extract the square root of $\dfrac{x^2y^2z^2}{25}$. *Ans.* $\pm\, 5^{-1}xyz$.

10. Extract the square root of $P^{2x}M^{4y}N^{-16}$. *Ans.* $\pm\, P^{x}M^{2y}N^{-8}$.

11. Extract the square root of $\dfrac{400}{a^{2m}b^{8n}c^2}$. *Ans.* $\pm\, 20a^{-m}b^{-4n}c^{-1}$.

12. Extract the square root of $\dfrac{a^{2m}b^{8n}c^2}{400}$ *Ans.* $\pm\, (20)^{-1}a^{m}b^{4n}c$.

13. Extract the square root of $a^{-2}b^{-2x}c^{-12z}$. *Ans.* $\pm\, a^{-1}b^{-1x}c^{-6z}$.

It is plain that a monomial will not be a perfect power when its coefficient is not a perfect square, and when every exponent is not some multiple of 2. But when the exponents of the literal factors are not exactly divisible by the index of the root, the division can be indicated. Thus, $\sqrt{4x} = \pm\, 2x^{\frac{1}{2}}$, for, by the rules for multiplication, $(\pm\, 2x^{\frac{1}{2}})(\pm\, 2x^{\frac{1}{2}}) = 4x$. Hence, $\pm\, 2x^{\frac{1}{2}}$ is, truly, the square root of $4x$. So, also, $\sqrt{ab^3c^5} = \pm\, a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{5}{2}}$, because $(\pm\, a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{5}{2}})\ (\pm\, a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{5}{2}}) = ab^3c^5$. The square roots, then, of all algebraic quantities may be truly expressed whenever their coefficients are perfect squares.

SQUARE ROOT OF POLYNOMIALS.

274. A trinomial is the least polynomial that is a perfect square. It would be a mistake to suppose that the square root of $a^2 + b^2$ is $a + b$, for $(a + b)^2 = a^2 + 2ab + b^2$. The term, $2ab$, enters into the square of $(a + b)$, and is not found in $a^2 + b^2$. Any polynomial, to be a perfect square, must be susceptible of resolution into two equal factors; and we know that when these factors are multiplied together to reproduce the polynomial, the two extreme terms of the product (if the

factors have been arranged with reference to a certain letter) are irreducible with the other terms. Hence, the square root of these extreme terms must be terms of the whole root. The extreme terms of $\sqrt{a^2+b^2}$ must then be a and b, but the intermediate terms can only be found approximatively.

Any binomial, as $(a'+s)$, when squared, will give a trinomial, $a'^2 + 2a's + s^2$. Conversely, if we have any trinomial that is a perfect square, its root must be a binomial.

Let it be required to extract the square root of $18ab + 81a^2 + b^2$. If this trinomial be arranged with reference to the highest power of one of its letters, as a, and the square root of the first term be taken, we know that we have certainly one term of the root required, Because, from what has been said, we know that the first term of the arranged trinomial is the product of the first terms of the two equal factors into which the trinomial can be resolved. In other words, it is the square of the first term of the required root. The arranged polynomial is $81a^2 + 18ab + b^2$. We begin by extracting the root of the first term, and set this root in the same horizontal line with the polynomial, and on its right separated by a vertical bar.

$$\begin{array}{r|l} 81a^2 + 18ab + b^2 & 9a + b \\ 81a^2 = a'^2 & \overline{18a + b = 2a's + s} \\ \hline 2a's + s^2 = 18ab + b^2 & \\ \underline{18ab + b^2} & \end{array}$$

We subtract from the given quantity the square of the first term of the root; there remain, then, only the two terms, $18ab + b^2$. Now, we know that the first term of this remainder, $18ab$, is the double product of the first term of the root by the unknown second term. If, then, we divide $18ab$ by $2\ (9a)$, or $18a$, the quotient, b, must be the second term of the root. The root is then completely known. The result can be verified by squaring the root $9a + b$; or, since the remainder $18ab + b^2$ corresponds to $2a's + s^2$, and since $18a$ corresponds to $2a'$, if we write $18a$ below the root, and b, which corresponds to s, on its right, and then multiply $18a + b$ by b, we must evidently form the remainder, $2a's + s^2$.

275. Since, when a trinomial is a perfect square, its extreme terms must be perfect squares, and its mean term must be the double product of the roots of these terms, we can tell in a moment when a trinomial is a perfect square; the mean term of the arranged trinomial must always be the double product of the roots of the extreme terms. Let us apply

this simple test to some expressions. $4a^2 + 4ma + m^2$ is a perfect square, and its root $2a + m$; $a + 2\sqrt{ab} + b$ is a perfect square, since the test is satisfied, and the root, $\sqrt{a} + \sqrt{b}$; $x^3 + 2x^{\frac{3}{2}}y^{\frac{3}{2}} + y^3$ is a perfect square, and its root, $x^{\frac{3}{2}} + y^{\frac{3}{2}}$.

The root, however, will only be commensurable when the extreme terms of the arranged trinomial are rational. We have, from the foregoing, a simple rule for the extraction of the root of a trinomial that is a perfect square.

RULE.

Extract the root of the extreme terms, and connect their roots together by the sign of the mean term.

Thus, the square root of $m^2 - 2mn + n^2$ is $m - n$; this is obvious, since $(m - n)^2 = m^2 - 2mn + n^2$; or it may be seen by going through the steps of the process described.

EXAMPLES.

1. Required $\sqrt{49a^2 + 14am + m^2}$. *Ans.* $7a + m$.
2. Required $\sqrt{49a^2 - 14am + m^2}$. *Ans.* $7a - m$.
3. Required $\sqrt{49a^2 + 28am + 4m^2}$. *Ans.* $7a + 2m$.
4. Required $\sqrt{49a^2 - 28am + 4m^2}$. *Ans.* $7a - 2m$.
5. Required $\sqrt{4m + 16\sqrt{mn} + 16n}$. *Ans.* $2\sqrt{m} + 4\sqrt{n}$.
6. Required $\sqrt{4m - 16\sqrt{mn} + 16n}$. *Ans.* $2\sqrt{m} - 4\sqrt{n}$.
7. Required $\sqrt{m^2 - 14am + 49a^2}$. *Ans.* $m - 7a$.
8. Required $\sqrt{4m^2 - 28am + 49a^2}$. *Ans.* $2m - 7a$.

Remarks.

Examples 2 and 4, in connection with 7 and 8, show, that when the mean term of the trinomial is negative there will be two distinct roots, according to the arrangement of the terms. The reason of this is plain. It is evident, moreover, that when either of the extreme terms has the negative sign, or when both are negative, the root will be imaginary.

The rule for the extraction of the square root of a trinomial has an important application in the solution of complete equations of the second degree, and ought, therefore, to be remembered.

276. *If the given number, instead of being a trinomial, have a trinomial root.*

Let $a + m + n$ represent this root. Then, by representing $a + m$ by p, and the given polynomial by N, we have $N = (p + n)^2 = p^2 + 2pn + n^2 = a^2 + 2am + m^2 + 2(a + m)n + n^2$. We see that the first three terms is a perfect square, and the root may be found by the rule for the extraction of the root of a trinomial; then, when we have taken the square of the root, $a^2 + 2am + m^2$, from the given polynomial, there will remain $2(a + m)n + n^2$. The first term of this remainder, $2an$, divided by $2a$, the double of the first term of the root will give n, the third term of the root. The remainder, $2(a + m)n + n^2$, may be put under the form $(2a + 2m + n)n$. If, then, the first two terms of the root found be doubled, and the third term added to them, and their sum be multiplied by the third term, the product thus formed will be equal to the remainder. The process for extracting the root of a polynomial which is the square of a trinomial, is precisely like that for extracting the root of a polynomial which is the square of a binomial. The divisor, to get any term of the root after the first, is twice the first term of the root; the terms of the root preceding the last term found are doubled, the last term added, and the product of the whole, by the last term, subtracted from the successive remainders.

The different steps can be best exhibited by the following example:

$$
\begin{array}{r|l}
4n^2+12mn+9m^2+4nb+6mb+b^2 & 2n+3m+b \\
a^2 = 4n^2 & \overline{4n + 3m} \\
\hline
2am+m^2+2(a+m)n+n^2=12mn+9m^2+4nb+6mb+b^2 & 4n+6m+b \\
2am+m^2=12mn+9m^2 & \\
\hline
2(a+m)n+n^2=4nb+6mb+b^2 & \\
(2a+2m+n)n=4nb+6mb+b^2 & \\
\hline
\end{array}
$$

We began by extracting the root of the first term, the root found was set on the right, and its square subtracted from the given polynomial. We had, then, taken out a^2 of the formula from the given expressions; we next doubled the root found, and used this double root ($2a$ of the formula) as a divisor to find the second term. This, when found (m of the formula), was set on the right of $2n$, and connected by its appropriate sign. The double of the first term, and the second term, written directly under the root, were next multiplied by the second term. We thus formed $2am + m^2$ of the formula, which, when subtracted from the first remainder, left $4nb + 6mb + b^2$, corresponding to $2(a + m)n + n^2$ of the formula. The first term of this remainder ($2an$ of the formula), divided by twice the first term ($2a$ of the formula), obviously, gave the third term (n) of the formula. The first two terms of the root were next doubled, and the third term added to their sum. Having thus formed $2a + 2m + n$ of the formula, we multiplied this sum, $4n + 6m + b$, by the third term, b, (or n of the formula). We thus plainly formed the three parts of which the remainder was composed; and the product being exactly equal to the last remainder, the polynomial had an exact root, $2n + 3m + b$.

The foregoing reasoning can be readily extended to a polynomial whose root contains four terms. For, let N represent the polynomial, l, the sum of the first three terms of the root, and n the last term of root. Then, $N = (l + n)^2 = l^2 + 2ln + n^2$. Suppose the first three terms to be a, b and c; then, $N = (a + b + c)^2 + 2(a + b + c)n + n^2$. Now, the first three terms is the square of a trinomial, and the root can, therefore, be extracted precisely as in the foregoing example. After the square of the root has been subtracted from the given polynomial there will remain $2(a + b + c)n + n^2$. The divisor to find n is, obviously, still $2a$, twice the first term of the root. And, since the remainder can be placed under the form of $|2(a + b + c) + n|n$, it is plain that, if the last term be added to twice the sum of the first three terms, and the whole sum be multiplied by the last term, we will form the three parts of the remainder. The process for extracting the root of a polynomial which is the square of four terms, is then identical with that for extracting the root of a polynomial which is the square of three terms, and that, we have seen, is the same as the process for extracting the root of a polynomial that is the square of a binomial. Now, if the root is composed of five terms, after the square of the first four terms has been subtracted from the given polynomial, the remainder will be found to be the double product of the four terms found by the unknown fifth term, and the divisor to find this term will still be twice the first term of the root. And so the reasoning may be extended to a polynomial whose root is made up of six, seven, or any number of terms. Hence, for extracting the root of any polynomial, we have the following

RULE.

Arrange the polynomial with reference to one of its letters. Extract the root of the first term, and set the root in the same horizontal line with the given polynomial, and on its right, separated from it by a vertical bar. Subtract the square of the root found from the given polynomial, and bring down the remaining terms for the first remainder. Write double the first term of the root found immediately beneath the place of the root, divide the first term of the first remainder by it, and write the quotient, which is the second term of the root, on the right of the first term of the root, and also beneath and on the right of double this term. Multiply the double of the first term, and the second term, itself affected with its proper sign, by the second term, and subtract the binomial product from this first remainder, and bring down the remaining terms for the second remainder. Divide the first term of the second remainder by the double of the first term of the root, and the quotient, affected with its proper sign, will be the third term of the root. Set this third term in the root, and also in another horizontal line on the right of double the sum of the first two terms of the root. Multiply the three terms in this horizontal line by the third term, and subtract their product from the second remainder. Continue this process until the final remainder is zero, the root will then be exact; or continue until the letter, according to which the polynomial has been arranged, has disappeared from one of the remainders. Then, if all the exponents of that letter in the given polynomial are positive, we conclude that the polynomial is not a perfect square.

Required the square root of $4x^2 + 12xy + 4xz + 16xl + 9y^2 + 6yz + 24yl + 8lz + 16l^2$.

The polynomial is already arranged.

	$4x^2+12xy+4xz+16xl+9y^2+6yz+24yl+8lz+16l^2+z^2$	$2x$
	$4x^2$	$4x+3y$
1st Rem.	$=12xy+4xz+16xl+9y^2+6yz+24yl+8lz+16l^2+z^2$	$4x+6y+z$
	$12xy \quad\quad +9y^2$	$4x+6y+2z+4l$
2d Rem.	$=4xz+16xl \quad +6yz+24yl+8lz+16l^2+z^2$	
	$4xz \quad\quad +6yz \quad\quad z^2$	
3d Rem.	$=16xl \quad\quad +24yl+8lz+16l^2$	
	$16xl \quad\quad +24yl+8lz+16l^2$	
	$0 \quad\quad 0 \quad 0 \quad 0$	

277. After the beginner has become familiar with the preceding principles, it will not always be necessary to go through the process of forming the successive remainders. The successive products, distinguished by parentheses, may be all formed, and their sum taken at once from the given polynomial, as in the following example.

$$\begin{array}{l|l} m^2 - 6mn + 9n^2 + 4mp - 2mq - 12np + 6nq + 4p^2 - 4pq + q^2 & m - 3n + 2p - q \\ \hline (m^2) + {}^2(-6mn + 9n^2) + {}^3(4mp - 12np + 4p^2) + {}^4(-2mq + 6nq - 4pq + q^2) & 2m - 3n \\ \hline 0 \quad\quad 0 \quad\quad\quad\quad 0 \quad\quad\quad\quad\quad\quad 0 & 2m - 6n + 2p \\ \hline & 2m - 6n + 4p - q \end{array}$$

We have distinguished the successive products by parentheses, affected by exponents written on the left.

278. It is obvious that the successive steps required by the general rule for extracting the square root of any polynomial, amount to nothing more than subtracting the square of the algebraic sum of the terms of the root, as they are found, from the given polynomial. In some cases, then, it may save trouble to subtract the square of the algebraic sum of the first two terms of the root from the given polynomial, then bring down the remaining terms and find the third term. Next, subtract the square of the algebraic sum of the first three terms of the root from the given polynomial, and continue in this way until all the terms of the root are found.

EXAMPLES.

1. Required $\sqrt{x^2 + 4y^2 - 6xz + 4xy - 12zy + 9z^2}$.

Ans. $x + 2y - 3z$.

2. Required $\sqrt{x^4 + y^4 - 2x^2y^2 + 4z^4 + 4z^2x^2 - 4z^2y^2}$.

Ans. $x^2 - y^2 + 2z^2$.

3. Required $\sqrt{m^2 + \frac{1}{4}n^2 - mn + \frac{x^2}{9} - \frac{1}{3}xn + \frac{2}{3}xm}$.

Ans. $m - \frac{1}{2}n + \frac{1}{3}x$.

4. Required $\sqrt{a^2 - 2ax + x^2 + 2am - 2an - 2xm + 2xn + m^2 - 2mn + n^2}$

Ans. $a - x + m - n$.

5. Required $\sqrt{x^6 + y^6 + 2x^3m + 2x^3y^3 - 2x^3n + 2y^3m - 2y^3n + m^2 - 2mn + n^2}$.

Ans. $x^3 + y^3 + m - n$.

6. Required $\sqrt{a^2+x^2+2ax+am+an+xn+xm+\frac{m^2}{4}+\frac{n^2}{4}+\frac{mn}{2}}$.

Ans. $a+x+\frac{m}{2}+\frac{n}{2}$.

7. Required $\sqrt{81x^2+9xy+\frac{y^2}{4}+9xz+9xm+\frac{yz}{2}+\frac{ym}{2}+\frac{z^2}{4}+\frac{zm}{2}+\frac{m^2}{4}}$.

Ans. $9x+\frac{y}{2}+\frac{z}{2}+\frac{m}{2}$.

8. Required $\sqrt{\frac{x^2}{4}+\frac{y^2}{4}+\frac{xy}{2}+6my+6mx+5zy+5zx+36m^2+25z^2+60mz}$.

Ans. $\frac{x}{2}+\frac{y}{2}+6m+5z$.

9. Required $\sqrt{y^2+4xy+4x^2+1+4x+2y}$.

Ans. $y+2x+1$.

10. Required $\sqrt{x^4+10x^2y^2+4x^3y+12xy^3+9y^4}$.

Ans $x^2+2xy+3y^2$.

11. Required $\sqrt{x^4+2x^3y+2x^4y+2x^3y^2+x^2y^2+x^4y^2}$.

Ans. x^2+xy+x^2y.

12. Required the square root of $x^4+10x^2y^2+4x^3y+12xy^3+9y^4+2x^3+6x^2y+10xy^2+6y^3+x^2+2xy+y^2$.

Ans. $x^2+2xy+3y^2+x+y$.

Remark.

The short process, indicated in Art. 277, cannot be followed in the last example.

SQUARE ROOT OF A POLYNOMIAL INVOLVING NEGATIVE EXPONENTS.

279. The principles for the extraction of the root of a polynomial containing negative exponents are the same as for the extraction of the square root of a polynomial, all of whose exponents are positive, observing, however, in the arrangement of the polynomial, that that negative exponent is the least algebraically which is the greatest numerically. When, too, the arranged letter has disappeared from any remainder, it

may be supplied with a zero exponent, *provided,* that the exponent of the same letter is negative in some of the terms.

Take, as an example, $x^{-2} + 2x^{-1}y + x^2y^2 + 2xy^2 + 2y + y^2$.

Arranging the polynomial with reference to the *ascending* powers of x, and proceeding as before, we have

$$\begin{array}{l|l}
x^{-2} + 2x^{-1}y + 2x^0y + x^0y^2 + 2xy^2 + x^2y^2 & x^{-1} + y + xy \\
x^{-2} & 2x^{-1} + y \\
\hline
\text{1st Rem.} = 2x^{-1}y + 2x^0y + x^0y^2 + 2xy^2 + x^2y^2 & 2x^{-1} + 2y + xy \\
2x^{-1}y + x^0y^2 & \\
\hline
\text{2d Rem.} = 2x^0y + 2xy^2 + x^2y^2 & \\
2x^0y + 2xy^2 + x^2y^2 &
\end{array}$$

The first term of the second remainder is $2y$; x, the letter according to which the polynomial was arranged, does not enter into this remainder until supplied. It, of course, must be introduced with a zero exponent, otherwise the expression $2y$ would be altered by its introduction. But, since $x^0 = 1$, then, $2x^0y = 2y$.

Take, as a second example, $x^{-4} + 2x^{-2}y + y^2 + 2 + 2yx^2 + x^4$.

$$\begin{array}{l|l}
x^{-4} + 2x^{-2}y + y^2 + 2 + 2yx^2 + x^4 & x^{-2} + y + x^2 \\
x^{-4} & 2x^{-2} + y \\
\hline
\text{1st Rem.} = 2x^{-2}y + y^2 + 2 + 2yx^2 + x^4 & 2x^{-2} + 2y + x^2 \\
2x^{-2}y + y^2 & \\
\hline
\text{2d Rem.} = 2x^0 + 2yx^2 + x^4 & \\
2x^0 + 2yx^2 + x^4 &
\end{array}$$

The first term of the second remainder is 2; x, affected with a zero exponent, was introduced into that term.

3. Required $\sqrt{x^{-6} + 2x^{-3}y^{-3} + y^{-6} + 2x^{-2} + 2xy^{-3} + x^2}$.

Ans. $x^{-3} + y^{-3} + x$.

4. Required the square root of $\frac{x^{-4}}{4} + \frac{x^{-2}y^{-3}}{3} + \frac{y^{-6}}{9} + 1 + \frac{2x^2y^{-3}}{3} + x^4$.

Ans. $\frac{x^{-2}}{2} + \frac{y^{-3}}{3} + x^2$.

5. Required the square root of $\frac{x^{-4}}{4} + \frac{x^{-2}y^{-3}}{3} + \frac{y^{-6}}{9} + 1 + \frac{2x^2y^{-3}}{3} + x^4 + 2x^{-2} + \frac{4y^{-3}}{3} + 4x^2 + 4$.

Ans. $\frac{x^{-2}}{2} + \frac{y^{-3}}{3} + 2 + x^2$.

6. Required the square root of $x^6 - 2x^5y + x^4y^2 + 2x^3y^{-1} - 2x^2 + 2 - 2x^{-1}y + 2x^{-3}y^{-1} + x^{-6} + y^{-2}$. *Ans.* $x^3 - x^2y + x^{-3} + y^{-1}$.

7. Required the square root of $x^{-8} + 2y + x^8y^2 + \frac{x^{-8}y^{-2}}{4} + x^{-8}y^{-1} + 1$.

Ans. $x^{-4} + \frac{x^{-4}y^{-1}}{2} + x^4y$.

INCOMMENSURABLE POLYNOMIALS.

280. When the exponent of the letter, according to which the polynomial is arranged, is positive in all the terms, we know that the polynomial is incommensurable when this letter is not found in any of the successive remainders, or is found affected with a lower exponent than it appears with in the first term of the root. But, if the letter, according to which the arrangement is made, appear in some of the terms with a negative exponent, we can only tell that the given polynomial is not commensurable by observing that the operation would never end.

EXAMPLES.

1. Required the square root of $a^2 + b^2$.

Ans. $a + \frac{b^2}{2a} - \frac{b^4}{8a^3} + \frac{b^6}{16a^5} - \&c.$

2. Required the square root of $1 - x^2$.

Ans. $1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \&c.$

3. Required the square root of $x^2 - 1$.

Ans. $x - \frac{1}{2x} - \frac{1}{8x^3} - \frac{1}{16x^5} + \&c.$

4. Required the square root of $2x^2 + 2x + 2$.

Ans. $x\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{3}{4x\sqrt{2}} + \&c.$, or $\sqrt{2}\,(x + \frac{1}{2} + \frac{3}{8x} + \&c.)$

5. Required the square root of $2x^2 - 6x + 4$.

Ans. $x\sqrt{2} - \frac{3}{\sqrt{2}} - \frac{1}{4x\sqrt{2}} - \&c.$, or $\sqrt{2}\,(x - \frac{3}{2} - \frac{1}{8x} - \&c.)$

6. Required the square root of $m + n$.

Ans. $\sqrt{m} + \frac{n}{2\sqrt{m}} - \frac{n^2}{8m\sqrt{m}} + \frac{n^3}{16m^2\sqrt{m}} - \&c.$

7. Required the square root of $x^{-4} + 2x^{-2} + 4x^{-1}$

Ans. $x^{-2} + 1 + 2x + \&c.$

SQUARE ROOT OF POLYNOMIALS CONTAINING TERMS AFFECTED WITH FRACTIONAL EXPONENTS.

281. Since, in multiplication, the exponents of like letters are added, whether they be fractional, or entire, it is evident that the square of $a^{\frac{m}{n}} = a^{\frac{m}{n}} \times a^{\frac{m}{n}}$ is $a^{\frac{2m}{n}}$. Hence, the square root of $a^{\frac{2m}{n}}$ is $a^{\frac{m}{n}}$. Quantities involving fractional exponents may then be operated upon in the same manner as quantities containing only entire exponents. This will be shown more fully under the head of fractional exponents. Assuming a truth which scarcely needs a demonstration, we will give a few examples of polynomials involving fractional exponents.

EXAMPLES.

1. Required $\sqrt{x^{10} + y^{10} - 2x^5y^5 - 2y^5z^{\frac{1}{2}} + 2x^5z^{\frac{1}{2}} + z}$.
   Ans. $x^5 - y^5 + z^{\frac{1}{2}}$.

2. Required $\sqrt{\frac{x}{4} + \frac{y}{4} + \frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{2} + \frac{z}{4} + \frac{x^{\frac{1}{2}}z^{\frac{1}{2}}}{2} + \frac{y^{\frac{1}{2}}z^{\frac{1}{2}}}{2}}$.
   Ans. $\frac{x^{\frac{1}{2}}}{2} + \frac{y^{\frac{1}{2}}}{2} + \frac{z^{\frac{1}{2}}}{2}$.

3. Required $\sqrt{\frac{x}{9} + \frac{y}{9} + \frac{2x^{\frac{1}{2}}y^{\frac{1}{2}}}{9} + \frac{z}{9} + \frac{2x^{\frac{1}{2}}z^{\frac{1}{2}}}{9} + \frac{2y^{\frac{1}{2}}z^{\frac{1}{2}}}{9}}$.
   Ans. $\frac{x^{\frac{1}{2}}}{3} + \frac{y^{\frac{1}{2}}}{3} + \frac{z^{\frac{1}{2}}}{3}$.

4. Required $\sqrt{x^{\frac{2}{5}} + y^{\frac{1}{5}} + 2x^{\frac{1}{5}}y^{\frac{1}{10}} + 2y^{\frac{1}{10}}a^2 + 2x^{\frac{1}{5}}a^2 + a^4}$.
   Ans. $x^{\frac{1}{5}} + y^{\frac{1}{10}} + a^2$.

CUBE ROOT OF POLYNOMIALS.

282. The cube of a monomial, as a, is a^3, for $(a)^3 = a \times a \times a = a^3$. The cube of a binomial, $a + b$, is $a^3 + 3a^2b + 3ab^2 + b^3$.

For, by actual multiplication, we have $(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$. Since, then, the cube of a monomial is a monomial, and the cube of a binomial a polynomial of four terms, it follows that a polynomial of four terms is the least polynomial which can have an exact cube root.

Knowing the third power of a binomial, we have only to reverse the process to obtain the cube root of the power. We see that the first term of the power is the perfect cube of a, and we know, from the manner in which a product is formed, that that term, a^3, has been derived without reduction from the multiplication of the three factors of which the power is composed. If then we extract the cube root of a^3, we know certainly that the root, a, is a term of the required root. Then a will be one of the extreme terms of the root; and, since the extreme terms of the root when cubed give powers that are irreducible with the other terms, it follows that the cube of a or a^3 is to be taken from the given polynomial. After this subtraction, the remainder is $3a^2b + 3ab^2 + b^3$, and it is plain that the second term of the root can be found by dividing the first term of the *arranged* remainder by $3a^2$, or by three times the square of the first term of the root. The remainder, $3a^2b + 3ab^2 + b^3$ can be put under the form of $(3a^2 + 3ab + b^2)b$. If then we add three times the square of the first term of the root, three times the first power of this term by the second term of the root, and the square power of the second term of the root, together, and multiply the sum of the three terms by the last term of the root, we will obviously form the three parts of which the remainder is composed. The process, then, for extracting the cube root, is analogous to that for extracting the square root of a polynomial.

$$
\begin{array}{lr|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & a + b \\
 & a^3 \qquad\qquad\qquad\qquad & \overline{3a^2 + 3ab + b^2} \\
\text{1st Rem.} & = 3a^2b + 3ab^2 + b^3 & \\
 & 3a^2b + 3ab^2 + b^3 & \\
\text{2d Rem.} & = 0 \qquad 0 \qquad 0 &
\end{array}
$$

The root is set in the same horizontal line with the given polynomial, the cube of the first term is taken from the given polynomial, and the first term of the arranged remainder is divided by three times the square of the first term of the root, to obtain the second term of the root. Finally, immediately beneath the root is written, three times the square of the first term of the root, plus three times the product of the first and second terms of the root, plus the square of the second term of the root, and the product of the sum of these three terms by the last term of the root is taken from the first remainder.

If the root be composed of three terms, a, b and c, we may represent the algebraic sum of a and b by m. Then, let P be the polynomial whose root is $a + b + c$, we will have $P = (a + b + c)^3 = (m + c)^3 = m^3 + 3m^2c + 3mc^2 + c^3 = (a + b)^3 + 3(a + b)^2 c + 3(a + b) c^2 + c^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + c^3$.

The first four terms of this polynomial are the same as in the last example, and therefore their root, $a + b$, can be found as in that example. After the cube of the sum of these terms has been taken from the given polynomial, the remainder may be placed under the form of $\mid 3(a + b)^2 + 3(a + b)c + c^2 \mid c$. Then, having found the third term by dividing the first term, $3a^2c$, of the arranged remainder, by $3a^2$, it is plain that the remainder itself can be formed as before, by multiplying the last term found into three times the square of the sum of the other terms of the root, plus three times the first power of the sum of the other terms into the last term, plus the square of the last term.

Now, whatever may be the number of terms in the root, we may represent the algebraic sum of all of them, except the last, by s. Suppose n to be the last term. Then $P = (s + n)^3 = s^3 + 3s^2n + 3sn^2 + n^3$: in which s represents the algebraic sum of any number of terms. If a is the first term of s, then the first term of the development of $3s^2n$, will be $3a^2n$. Hence, the divisor to find n is still three times the square of the first term. After s^3 has been subtracted from the polynomial, the remainder can be put under the form of $(3s^2 + 3sn + n^2)n$. Hence, whatever the number of terms in the root, the successive remainders can be formed just as when there are but two or three terms in the root.

The following examples will illustrate the process.

$$\begin{array}{ll|l}
 & x^6+3x^5+3x^4+9x^4y+x^3+18x^3y+27x^2y^2+9x^2y+27xy^2+27y^3 & x^2+x+3y \\
 & x^6 & (3x^4+3x^3+x^2)x \\
\text{1st R.} & =3x^5+3x^4+9x^4y+x^3+18x^3y+27x^2y^2+9x^2y+27xy^2+27y^3 & (3(x^2+x)^2+3(x^2+x)3y+9y^2)3y \\
 & 3x^5+3x^4 \qquad +x^3 & (3x^4+6x^3+3x^2+9x^2y+9yx+9y^2)3y \\
\text{2d Rem.} & =9x^4y+18x^3y+27x^2y^2+9x^2y+27xy^2+27y^3 & \\
 & 9x^4y+18x^3y+27x^2y^2+9x^2y+27xy^2+27y^3 & \\
\text{3d Rem.} & =\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 &
\end{array}$$

$$\begin{array}{ll|l}
 & x^6+6x^5+15x^4+20x^3+15x^2+6x+1 & x^2+2x+1 \\
 & x^6 & (3x^4+6x^3+4x^2)2x \\
\text{1st R.} & =6x^5+15x^4+20x^3+15x^2+6x+1 & (3(x^2+2x)^2+3(x^2+2x)1+1)1= \\
 & 6x^5+12x^4+\ 8x^3 & (3x^4+12x^3+12x^2+3x^2+6x+1)1 \\
\text{2d Rem.} & =3x^4+12x^3+15x^2+6x+1 & \\
 & 3x^4+12x^3+15x^2+6x+1 & \\
\text{3d Rem.} & =\ 0 \quad 0 \quad 0 \quad 0 \quad 0 &
\end{array}$$

RULE.

Arrange the given polynomial with reference to the ascending or descending powers of one of its letters. Extract the cube root of the term on the left, and set the root on the same horizontal line with the given polynomial on the right, and separated by a vertical line. Subtract the cube of this first term of the root from the given polynomial, and bring down the remaining terms for the first remainder. Write three times the square of the first term of the root immediately beneath the place of the root. This will be the divisor to find all the other terms of the root. Divide the first term of the remainder by the divisor, and set the quotient, with its appropriate sign, on the right of the first term of the root, for the second term of the root. Set, on the right of the divisor, three times the product of the first and second terms of the root, affected with its proper sign, plus the square of the second term of the root. Next, multiply the algebraic sum of the three terms, thus formed, by the last term of the root, and subtract the product from the first remainder. Bring down the remaining terms for the second remainder, and divide the first term by the divisor. The quotient is the third term of the root. Take three times the square of the sum of the first two terms, and add to this three times the product of the algebraic sum of the first two terms by the third term, plus the square of the third term. Multiply the algebraic sum of the whole by the third term, and subtract the product from the second remainder. Bring down the remaining terms from the third remainder. Find the fourth term of the root as before, and continue the process until all the terms of the root are found.

Remarks.

1st. When the exponent of the letter, according to which the arrangement is made, is positive in all the terms of the given polynomial, we know that the root is not exact, whenever the exponent of the assumed letter is less in the first term of any of the successive arranged remainders than it is in the divisor.

2d. It is evident that the steps of the process, according to the rule, amount to nothing more than subtracting the cube of the algebraic sum of the terms of the root, when found, from the given polynomial. It may, therefore, sometimes save trouble to proceed thus. Subtract the cube of the first term from the given polynomial. Find the second term of the root according to the rule. Cube the algebraic sum of the two terms of the root, and subtract the result from the given polynomial. Find the third term, and subtract the cube of the algebraic sum of the first three terms of the root from the given polynomial. Proceed in this manner until we get a zero remainder, or until it becomes evident that the given polynomial is incommensurable.

EXAMPLES.

1. Required the cube root of $x^9 + 3x^8 + 3x^7 + 7x^6 + 12x^5 + 6x^4 + 12x^3 + 12x^2 + 8$. *Ans.* x^3+x^2+2.

2. Required the cube root of $x^6 + 6x^5 + 21x^4 + 44x^3 + 63x^2 + 54x + 27$. *Ans.* $x^2 + 2x + 3$.

3. Required the cube root of $8x^6 + 48x^5 + 168x^4 + 352x^3 + 504x^2 + 432x + 216$. *Ans* $2x^2 + 4x + 6$.

4. Required the cube root of $\frac{x^6}{8} + \frac{x^5}{4} + \frac{11x^4}{12} + \frac{28x^3}{27} + \frac{11x^2}{6} + x + 1$.

Ans. $\frac{x^2}{2} + \frac{x}{3} + 1$

5. Required the cube root of $\frac{125x^9}{64} + \frac{75x^7}{64} + \frac{15x^5}{64} + \frac{x^3}{64}$.

Ans. $\frac{5x^3}{4} + \frac{1x}{4}$.

6. Required the cube root of $\frac{125x^9}{8} + \frac{75x^7}{8} + \frac{15x^5}{8} + \frac{x^3}{8}$.

Ans. $\frac{5x^3}{2} + \frac{1x}{2}$.

7. Required the cube root of $\frac{x^9}{8} + \frac{3x^8}{8} + \frac{3x^7}{8} + \frac{7x^6}{8} + \frac{3x^5}{2} + \frac{3x^4}{4} + \frac{3x^3}{2} + \frac{3x^2}{2} + 1$. *Ans.* $\frac{x^3}{2} + \frac{x^2}{2} + 1$.

8. Required the cube root of $8a^3 + 12a^2b - 12a^2m + 6ab^2 - 12amb + 6am^2 - 3mb^2 + 3m^2b + b^3 - m^3$ *Ans.* $2a + b - m$.

9. Required the cube root of $x^3 - 9x^2y - 3x^2 + 27xy^2 + 18xy + 3x - 27y^3 - 27y^2 - 9y^1 - 1$. *Ans.* $x - 3y - 1$.

10. Required the cube root of $x^6 + 3x^5 + 3x^4y^2 - 3x^4y + 3x^4 + x^3 + 6x^3y^2 - 6x^3y + 6x^2y^2 + 3x^2y^4 - 6x^2y^3 - 3x^2y + 3xy^2 - 6xy^3 + 3xy^4 + y^6 - 3y^5 + 3y^4 - y^3$. *Ans.* $x^2 + x + y^2 - y$.

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CUBE ROOT OF INCOMMENSURABLE POLYNOMIALS.

283. — EXAMPLES.

1. Required the cube root of $x^3 + 3x^2 + 3x + 3$.

$$Ans.\ x + 1 + \frac{2}{3x^2} +, \&c.$$

2. Required the cube root of $x^3 + 6x^2 + 12x + 9$.

$$Ans.\ x + 2 + \frac{1}{3x^2} +, \&c.$$

3. Required the cube root of $x^3 + a^3$.

$$Ans.\ x + \frac{a^3}{3x^2} - \frac{a^6}{9x^5}, \&c.$$

4. Required the cube root of $x^3 + 3x^2 + 4$.

$$Ans.\ x + 1 - \frac{1}{x} -, \&c.$$

5. Required the cube root of $x + 3x^2 + 6$.

$$Ans.\ \sqrt[3]{x} + \frac{x^2}{\sqrt[3]{x^2}} +, \&c.$$

6. Required the cube root of $x^{-3} + 3x^{-2} + 3x^{-1} + 2$.

$$Ans.\ x^{-1} + 1 + \frac{1}{3x^{\ 2}} +, \&c.$$

7. Required the cube root of $x^{-3} + 6x^{-2} + 12x^{-1} + 9$.

$$Ans.\ x^{\ 1} + 2 + \frac{1}{3x^{-2}} +, \&c.$$

8. Required the cube root of $x^{-6} + 3x^{-4} + 3x^{-2} + 2$.

$$Ans.\ x^{-2} + 1 + \frac{1}{3x^{-4}} +, \&c.$$

9. Required the cube root of $x^9 + 9x^6 + 27x^3 + 26$.

$$Ans.\ x^3 + 3 - \frac{1}{3x^6} -, \&c.$$

10. Required the cube root of $x^6 + 3x^5 + 3x^4 + x^2$.

$$Ans.\ x^2 + x - \frac{1}{3x} -, \&c.$$

CUBE ROOT OF POLYNOMIALS INVOLVING FRACTIONAL EXPONENTS.

284. — EXAMPLES.

1. Required the cube root of $x^{\frac{3}{2}} + 3x + 3x^{\frac{1}{2}} + 1$.

Ans. $x^{\frac{1}{2}} + 1$.

2. Required the cube root of $x + 3x^{\frac{7}{9}} + 3x^{\frac{5}{9}} + x^{\frac{1}{3}}$

Ans. $x^{\frac{1}{3}} + x^{\frac{1}{9}}$.

3. Required the cube root of $x^6 + 3x^3 + x^{\frac{3}{2}} + 3x^{\frac{9}{2}}$.

Ans. $x^2 + x^{\frac{1}{2}}$.

4. Required the cube root of $x^{15} + 3x^{\frac{51}{5}} + 3x^{\frac{12}{5}} + x^{\frac{3}{5}}$.

Ans. $x^5 + x^{\frac{1}{5}}$.

5. Required the cube root of $x^6 + 3x^{\frac{9}{2}} + 3x^4 + 3x^3 + 6x^{\frac{5}{2}} + 3x^2 + x^{\frac{3}{2}} + 3x + 3x^{\frac{1}{2}} + 1$.

Ans. $x^2 + x^{\frac{1}{2}} + 1$.

6. Required the cube root of $8x^6 + 24x^{\frac{9}{2}} + 24x^4 + 48x^{\frac{5}{2}} + 24x^2 + 8x^{\frac{3}{2}} + 24x + 24x^{\frac{1}{2}} + 8$.

Ans. $2x^2 + 2x^{\frac{1}{2}} + 2$.

7. Required the cube root of $x^{\frac{1}{2}} + 3x^{\frac{4}{3}} + 3x^{\frac{13}{6}} + x^3$.

Ans. $x^{\frac{1}{6}} + x$.

8. Required the cube root of $x^{\frac{1}{3}} + 3x^{\frac{7}{6}} + 3x^2 + x^{\frac{17}{6}}$.

Ans. $x^{\frac{1}{9}} + x^{\frac{17}{18}}$.

9. Required the cube root of $x^2 + 3x^{\frac{16}{9}} + 3x^{\frac{14}{9}} + x^{\frac{4}{3}}$.

Ans. $x^{\frac{2}{3}} + x^{\frac{4}{9}}$.

10. Required the cube root of $x^{12} + 3x^{\frac{33}{4}} + 3x^{\frac{9}{2}} + 1 + 3x^8 + 3x^{\frac{1}{2}} + 6x^{\frac{17}{4}} + 3x^4 + 3x^{\frac{1}{4}} + x^{\frac{3}{4}}$.

Ans. $x^4 + x^{\frac{1}{4}} + 1$.

11. Required the cube root of $x^{-12} + 3x^{-\frac{33}{4}} + 3x^{-\frac{9}{2}} + 1 + 3x^{8} + 6x^{-\frac{17}{4}} + 3x^{-\frac{1}{2}} + 3x^{-4} + x^{-\frac{3}{4}} + 3x^{-\frac{1}{4}}$.

Ans. $x^{-4} + x^{-\frac{1}{4}} + 1$.

12. Required the cube root of $x^{-15} + 3x^{-\frac{51}{5}} + 3x^{-\frac{12}{5}} + x^{-\frac{3}{5}}$.

Ans. $x^{-5} + x^{-\frac{1}{5}}$.

13. Required the cube root of $x^6 - 3x^{\frac{13}{3}} + 3x^{\frac{8}{3}} - x$.

Ans. $x^2 - x^{\frac{1}{3}}$.

14. Required the cube root of $x^6 + 3x^5 + 6x^4 + 3x^{\frac{9}{2}} + 6x^{\frac{7}{2}} + 9x^{\frac{5}{2}} + 10x^3 + 9x^2 + 7x^{\frac{3}{2}} + 6x + 3x^{\frac{1}{2}} + 1$. *Ans.* $x^2 + x + x^{\frac{1}{2}} + 1$.

15. Required the cube root of $1 + 3x^{-1} + 6x^{-2} + 3x^{-\frac{3}{2}} + 6\ ^{-\frac{5}{2}} + 9x^{-\frac{7}{2}} + 10x^{-3} + 9x^{-4} + 7x^{-\frac{9}{2}} + 6x^{-5} + 3x^{-\frac{11}{2}} + x^{-6}$.

Ans. $1 + x^{-1} + x^{-\frac{3}{2}} + x^{-2}$.

16. Required the cube root of $27x^6 + 81x^5 + 162x^4 + 81x^{\frac{9}{2}} + 162x^{\frac{7}{2}} + 243x^{\frac{5}{2}} + 270x^3 + 243x^2 + 189x^{\frac{3}{2}} + 162x + 81x^{\frac{1}{2}} + 27$.

Ans. $3\,(x^2 + x + x^{\frac{1}{2}} + 1)$.

CUBE ROOT OF POLYNOMIALS CONTAINING ONE OR MORE TERMS AFFECTED WITH NEGATIVE EXPONENTS.

285. The process is the same as for polynomials, all of whose terms are affected with positive exponents; observing, in the arrangement, that that negative exponent is the least algebraically which is the greatest numerically, and that when the letter, according to which the arrangement has been made, has disappeared from any of the successive remainders, it may be introduced affected with a zero exponent.

EXAMPLES.

1. Required the cube root of $x^{-6} + 3 + 3x^6 + x^8$.

Ans. $x^{-2} + x^4$.

For, by the rule

$$\begin{array}{rl|l}
 & x^{-6} + 3 + 3x^6 + x^{12} & x^{-2} + x^4 \\
 & x^{-6} & (3x^{-4} + 3x^2 + x^8)x^4 \\
\hline
\text{1st Rem.} = & 3x^0 + 3x^6 + x^{12} & \\
 & 3x^0 + 3x^6 + x^{12} & \\
\hline
\text{2d Rem.} = & 0 \quad 0 \quad 0 &
\end{array}$$

This polynomial might have been arranged with reference to the descending powers of x.

2. Required the cube root of $x^{-6} + 3x^{-3} + 3 + x^3$.
Ans. $x^{-2} + x$.

3. Required the cube root of $x^{-6} + 3x^{-4} + 3x^{-3} + 3x^{-2} + 6x^{-1} + 4 + 3x + 3x^2 + x^3$. *Ans.* $x^{-2} + x + 1$.

4. Required the cube root of $x^3 + 3x^2 + 3x + 7 + 12x^{-1} + 6x^{-2} + 12x^{-3} + 12x^{-4} + 8x^{-6}$. *Ans.* $x + 1 + 2x^{-2}$.

5. Required the cube root of $1 + 6x^{-1} + 21x^{-2} + 44x^{-3} + 63x^{-4} + 54x^{-5} + 27x^{-6}$. *Ans.* $1 + 2x^{-1} + 3x^{-2}$.

6. Required the cube root of $a^{-3} — 6a^{-4}b + 12a^{-5}b^2 —8a^{-6}b^3$.
Ans. $a^{-1} — 2a^{-2}b$.

7. Required the cube root of $x^3 + 6x^2 — 40 + 96x^{-2} — 64x^{-3}$.
Ans. $x + 2 — 4x^{-1}$.

8. Required the cube root of $1 + 6x^{-1} — 40x^{-3} + 96x^{-5} — 64x^{-6}$.
Ans. $1 + 2x^{-1} — 4x^{-2}$.

9. Required the cube root of $1 — 6x^{-1} + 15x^{-2} — 20x^{-3} + 15x^{-4} + — 6x^{-5} + x^{-6}$. *Ans.* $1 — 2x^{-1} + x^{-2}$.

QUANTITIES AFFECTED WITH FRACTIONAL EXPONENTS.

286. Suppose it be required to multiply $a^{\frac{1}{2}}$ into itself until it is taken three times as a factor. Then, by the rules for multiplication, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{3}{2}}$. This result is, evidently, the cube of $a^{\frac{1}{2}}$. Now, $a^{\frac{1}{2}}$, multiplied by itself once, gives $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{2}{2}} = a$; and $\sqrt{a}$, multiplied by itself once, gives $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$. So, $a^{\frac{1}{2}}$, and $\sqrt{a}$, are equivalent expressions. The expression, $a^{\frac{3}{2}}$, then, which indicates the cube of $a^{\frac{1}{2}}$, also indicates the cube of $\sqrt{a}$. The numerator of the fractional exponent denotes the power to which the quantity has been raised, and the denominator the degree of the root to be extracted. Taking $a^{\frac{1}{3}}$ three times as a factor, we have $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$. Now,

the $\sqrt[3]{a}$, raised to the third power, will also plainly be a, because, from the definition of Involution and Evolution, it is evident that the latter undoes what the former has done. The root of any quantity raised to a power indicated by the index of the root, must then be the quantity itself. Hence, $(\sqrt[3]{a})^3 = a$; then, $a^{\frac{1}{3}} = \sqrt[3]{a}$; since we have seen that both $a^{\frac{1}{3}}$, and $\sqrt[3]{a}$, when cubed, give a. In general, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. Because $a^{\frac{m}{n}}$, taken as a factor n times, gives $a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}} = a^m$, and $\sqrt[n]{a^m}$, raised to the n^{th} power, is also a^m. The fractional index may be regarded as denoting *a power of a root.* The denominator expresses the root, and the numerator the power. The denominator shows into how many equal factors, or roots, the given quantity is divided, and the numerator shows how many of these factors have been taken. The fractional index may be considered to denote *the root of a power*, as well as *the power of a root.* Thus, $a^{\frac{3}{2}}$ may either denote that the third power of the $\sqrt{a}$ has been formed, or that the square root of the third power has to be taken. Taking the former view, it indicates an executed operation; taking the latter view, an unexecuted operation.

In general, $a^{\frac{m}{n}}$ may be read the m^{th} power of the n^{th} root of a, or the n^{th} root of the m^{th} power of a. It is more convenient to read the expression thus, a to the m divided by n, power; or a, raised to a power denoted by the quotient of m, divided by n. So, $a^{\frac{p}{q}}$ is read, a to the $\frac{p}{q}$ power.

There are two consequences of notation by means of fractional exponents that deserve consideration. 1st. Any multiple of the numerator of a fractional exponent may be taken, provided an equal multiple of the denominator is also taken. Thus, $a = a^{\frac{1}{1}}$ is also $= a^{\frac{3}{3}} = a^{\frac{8}{8}} = a^{\frac{m}{m}}$, &c. So, also, $a^{\frac{p}{q}} = a^{\frac{rp}{rq}}$. The reason of this is plain; the increase of the denominator makes the equal roots or factors of the given quantity, a, smaller; but the numerator being increased just as much as is the denominator, the number of these smaller equal roots will be increased enough to make up for their diminution in magnitude. 2d. Since the fractional exponent may be exchanged for any other of equal value, it may be expressed in decimals. Thus, $a^{\frac{1}{2}} = a^{\frac{.5}{1.0}} = a^{.5}$. So, also, $a^{\frac{1}{5}}$

$= a^{\frac{2}{10}} = a^{.2}$; $a^{\frac{3}{5}} = a^{.6}$; $a^{\frac{12}{5}} = a^{2.4}$; $a^{\frac{4}{3}} = a^{1.333+}$; $a^{\frac{1}{9}} = a^{.11111+}$; $a^{\frac{1}{27}}$ $= a^{.037037+}$.

These decimal indices are called *logarithms*. The manner of calculating them, and their use, will be shown hereafter.

MULTIPLICATION OF QUANTITIES AFFECTED WITH ENTIRE OR FRACTIONAL EXPONENTS. — MONOMIALS.

287. Quantitities with fractional exponents must plainly be multiplied in the same way as quantities affected with entire exponents. That is, the exponents of the same letter or letters in the multiplicand and multiplier must be added, and after the common letter or letters must be written those not common, with their primitive exponents.

Let it be required to multiply $a^{\frac{3}{2}}$ by a^2. Then, $a^{\frac{3}{2}}$ is to be repeated a^2, or $a^{\frac{4}{2}}$ times. Now, to repeat $a^{\frac{3}{2}}$, $a^{\frac{4}{2}}$ times, we have only to add the exponents of multiplicand and multiplier, for the first exponent denotes that $a^{\frac{1}{2}}$ has been taken three times to produce the product, $a^{\frac{3}{2}}$; and the second exponent denotes that $a^{\frac{1}{2}}$ has been taken four times in the product, $a^{\frac{4}{2}}$. Hence, $a^{\frac{1}{2}}$ must enter seven times in the result of the multiplication of $a^{\frac{3}{2}}$ by $a^{\frac{4}{2}}$, and that result must be written $a^{\frac{7}{2}}$. To multiply $a^{\frac{1}{2}}$ by b, is to repeat $a^{\frac{1}{2}}$, b times. Hence, $a^{\frac{1}{2}} \times b = a^{\frac{1}{2}}b$.

RULE.

Multiply the coefficients together for the coefficient of the product. Reduce the exponents of the same letters in multiplicand and multiplier to the same denominator, add their numerators, and set their sum over the common denominator. Annex these letters to the new coefficient, and write after them all the letters which are not common to the multiplicand and multiplier.

EXAMPLES.

1. Multiply $a^{\frac{1}{m}}b^{\frac{1}{n}}$ by $a^{\frac{2}{m}}b^{\frac{3}{n}}c$. *Ans.* $a^{\frac{3}{m}}b^{\frac{4}{n}}c$.

2. Multiply $\frac{a^{\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{8}}}{2}$ by $\frac{a^2b^3c^{+1}}{3}$. *Ans.* $\frac{a^{\frac{7}{3}}b^{\frac{13}{4}}c^{\frac{9}{8}}}{6}$.

3. Multiply $\dfrac{a^{\frac{2}{5}}b^{\frac{3}{4}}c^{\frac{1}{6}}}{m}$ by $\dfrac{a^{\frac{3}{4}}b^{\frac{2}{5}}c^{6}}{n}$. *Ans.* $\dfrac{a^{\frac{23}{20}}b^{\frac{23}{20}}c^{\frac{37}{6}}}{mn}$.

4. Multiply $\dfrac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}}$ by $a^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{3}{2}}z^{\frac{3}{2}}$. *Ans.* $axyz$.

5. Multiply $a^{-n}b^{-\frac{1}{m}}c^{-\frac{1}{p}}$ by $a^{n}b^{\frac{1}{m}}c^{\frac{1}{p}}$. *Ans.* 1.

6. Multiply $a^{-\frac{r}{s}}b^{-\frac{p}{q}}c^{-\frac{1}{2}}$ by $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{2}}$. *Ans.* $a^{\frac{s-2r}{2s}}b^{\frac{q-3p}{3q}}$.

7. Multiply $2a^{-\frac{2}{n}}b^{-\frac{3}{n}}c^{-\frac{4}{n}}$ by $\dfrac{a^{\frac{3}{n}}b^{\frac{4}{n}}c^{\frac{4}{n}}}{2}$. *Ans.* $a^{\frac{1}{n}}b^{\frac{1}{n}}$.

8. Multiply $8x^{\frac{1}{3}}y^{\frac{3}{4}}z^{\frac{1}{5}}$ by $-7x^{\frac{1}{2}}b^{\frac{1}{2}}z^{\frac{1}{2}}$. *Ans.* $-56x^{\frac{5}{6}}y^{\frac{3}{4}}z^{\frac{7}{10}}$.

9. Multiply $12a^{m}b^{n}c^{p}$ by $5a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$. *Ans.* $60a^{\frac{2m+1}{2}}b^{\frac{3n+1}{3}}c^{\frac{4p+1}{4}}$.

10. Multiply $4a^{-\frac{m}{n}}b^{-\frac{r}{s}}c^{-\frac{p}{q}}$ by $a^{\frac{m}{n}}b^{\frac{r}{s}}c^{\frac{p}{q}}$. *Ans.* 4.

DIVISION OF QUANTITIES AFFECTED WITH FRACTIONAL AND ENTIRE EXPONENTS.—MONOMIALS.

288. Since the exponents of the like letters are added in multiplication, they must be subtracted in division. For, the object of division is to find a quantity called the quotient, which, when multiplied by the divisor, will give the dividend. Let it be required to divide a^2 by $a^{\frac{1}{2}}$; then, we must find a quantity, which, when multiplied by $a^{\frac{1}{2}}$ will give a^2. This quantity is plainly $a^{\frac{3}{2}}$, for $a^{\frac{1}{2}} \times a^{\frac{3}{2}} = a^{\frac{1}{2}+\frac{3}{2}} = a^{\frac{4}{2}} = a^2$. But the quantity found, $a^{\frac{3}{2}}$, has resulted from the subtraction of the exponent of $a^{\frac{1}{2}}$ from $2 = \frac{4}{2}$, the exponent of a^2. So, the result of the division of $a^{\frac{1}{m}}$ by $a^{\frac{1}{n}}$ must be $a^{\frac{1}{m}-\frac{1}{n}}$, since $a^{\frac{1}{m}-\frac{1}{n}} \times a^{\frac{1}{n}} = a^{\frac{1}{m}}$. When the denominators of the fractional exponents are different, the subtraction can only be indicated, not performed, until they are reduced to the same index. If the exponents of the same letter in the divisor exceed that of the dividend, the exponent of that letter in the quotient

will be negative. So, also, if there are any letters in the divisor not common to the dividend, they must appear in the quotient, with their primitive exponents taken with a contrary sign; else, when the quotient and divisor are multiplied together, these letters would enter in the product.

RULE.

Divide the coefficient of the dividend by that of the divisor, the result will be the coefficient of the quotient. Write after it all letters common to dividend and divisor, affected with exponents equal to the difference of their exponents in the dividend and divisor; and, also, all the letters common to the dividend only, with their primitive exponents; and all common to the divisor only, with their primitive exponents taken with a contrary sign.

EXAMPLES.

1. Divide $4a^2c^{\frac{3}{2}}d^{\frac{1}{5}}$ by $2a^{\frac{1}{2}}c^{\frac{1}{2}}d^{\frac{1}{4}}$. *Ans.* $2a^{\frac{3}{2}}cd^{-\frac{1}{20}}$.

2. Divide $12a^{\frac{1}{r}}b^{\frac{1}{s}}c^{\frac{1}{4}}d^2$ by $6a^{\frac{1}{m}}b^2$. *Ans.* $2a^{\frac{m-r}{mr}}b^{\frac{1-2s}{2s}}c^{\frac{1}{4}}d^2$.

3. Divide $7x^my^nz^{\frac{1}{p}}$ by $x^my^nz^{-p}$. *Ans.* $7z^{\frac{1-p^2}{p}}$.

4. Divide $7z^{-\frac{1}{2}}y^{-\frac{1}{3}}$ by $z^{\frac{1}{2}}y^{\frac{2}{3}}x^{\frac{1}{4}}$. *Ans.* $7z^{-1}y^{-1}x^{-\frac{1}{4}}$.

5. Divide $24a^{\frac{1}{4}}b^{\frac{1}{5}}c^{\frac{1}{6}}$ by $24z^2a^{-\frac{3}{4}}b^{-\frac{4}{5}}c^{-\frac{5}{6}}$. *Ans.* $z^{-2}abc$.

6. Divide $18x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$ by $\dfrac{18}{x^{-\frac{1}{2}}y^{-\frac{1}{3}}z^{-\frac{1}{4}}}$. *Ans.* 1.

7. Divide $3a^{-p}b^{-\frac{1}{r}}c^2$ by $\frac{1}{3}a^{-p}b^{-\frac{2}{r}}d^{-\frac{1}{m}}$. *Ans.* $9b^{\frac{1}{r}}c^2d^{\frac{1}{m}}$.

8. Divide $80m^{\frac{1}{10}}b^{\frac{2}{3}}c^{\frac{3}{4}}$ by $5m^{\frac{1}{5}}b^{-\frac{1}{3}}x^{\frac{1}{2}}$. *Ans.* $16m^{-\frac{1}{10}}bc^{\frac{3}{4}}x^{-\frac{1}{2}}$.

9. Divide $360e^{-\frac{1}{m}}f^{\frac{1}{n}}g^{-\frac{1}{p}}$ by $180efg$. *Ans.* $2e^{-\frac{(m+1)}{m}}f^{-\frac{(1+n)}{n}}g^{-\frac{(1+p)}{p}}$.

10. Divide $4a^{\frac{1}{m}}c^{\frac{1}{3}}b^{\frac{1}{4}}$ by $a^{\frac{1}{m}}d^{-\frac{1}{3}}e^{-\frac{1}{4}}$. *Ans.* $c^{\frac{1}{3}}b^{\frac{1}{4}}d^{\frac{1}{3}}e^{\frac{1}{4}}$.

RAISING TO POWERS QUANTITIES AFFECTED WITH FRACTIONAL AND ENTIRE EXPONENTS.—MONOMIALS.

289. Since, to raise a quantity to a power is only to multiply it by itself a certain number of times, the rule for the involution of quantities affected with any exponents whatever is deduced directly from the rule of multiplication.

RULE.

Raise the coefficient to the required power, and write after it all the literal factors affected with exponents equal to the product of their primitive exponents by the exponent of the power.

Thus, to raise $a^{\frac{1}{2}}$ to the third power, is to take $a^{\frac{1}{2}}$ three times as a factor. Then, $(a^{\frac{1}{2}})^3 = a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = a^{\frac{3}{2}}$. The exponent of the result is the product arising from multiplying the exponent of the quantity by 3, the exponent of the power. So, $(\text{M}x^{\frac{1}{m}})^n = (\text{M})^n x^{\frac{m}{n}}$, because, $(\text{M}x^{\frac{1}{m}})^n = \text{M} \times \text{M} \times \text{M}x^{\frac{1}{m}+\frac{1}{m}+\frac{1}{m}} + \&c.$ If the given quantity be a fraction, it is raised to the required power by raising the numerator and denominator, separately, to that power: because, the power of a fraction is nothing more than the product of the fraction multiplied by itself a certain number of times; and a fraction is multiplied by itself by multiplying the numerators and denominators together separately.

EXAMPLES.

1. Find the $\frac{1}{r}$th power of a^m. *Ans.* $a^{\frac{m}{r}}$.

2. Find the mth power of $a^{\frac{1}{r}}$. *Ans.* $a^{\frac{m}{r}}$.

3. Find the $-\frac{1}{r}$th power of a^m. *Ans.* $a^{-\frac{m}{r}}$.

4. Find the $-m^{th}$ power of $a^{\frac{1}{r}}$. *Ans.* $a^{\frac{-m}{r}}$.

5. Find the 4^{th} power of $3a^2c^3b^4$. *Ans.* $81a^8c^{12}b^{16}$.

6. Find the 3^d power of the 4^{th} power of $\frac{2a}{b}$. *Ans.* $128a^7b^{-7}$.

7. Find the q^{th} power of $\frac{2a}{b}$. *Ans.* $(2)^q a^q b^{-q}$.

8. Find the $-5m^{th}$ power of a^2b^3. *Ans.* $a^{-10m}b^{-15m}$.

9. Find the $-5m^{th}$ power of the 3^d power of a^2b^3.
Ans. $a^{-30m}b^{-45m}$.

10. Find the -12^{th} power of the -6^{th} power of $-\frac{a}{b}$.
Ans. $+a^{72}b^{-72}$.

11. Find the -12^{th} power of the -7^{th} power of $-\frac{a}{b}$.
Ans. $+a^{84}b^{-84}$.

12. Find the -11^{th} power of the -7^{th} power of $-\frac{a}{b}$.
Ans. $-a^{77}b^{-77}$.

13. Find the r^{th} power of the m^{th} power of a^n. *Ans.* a^{nmr}.

14. Find the $-r^{th}$ power of the m^{th} power of a^{-n}. *Ans.* a^{nmr}.

15. Find the r^{th} power of the $-m^{th}$ power of a^{-n}. *Ans.* a^{nmr}.

16. Find the $-r^{th}$ power of the $-m^{th}$ power of a^n. *Ans.* a^{nmr}.

17. Find the 5^{th} power of $\frac{a^{\frac{1}{2}}b^{\frac{1}{3}}}{c}$. *Ans.* $\frac{a^{\frac{5}{2}}b^{\frac{5}{3}}}{c^5}$.

18. Find the 5^{th} power of $\frac{a^{\frac{1}{2}}b^{\frac{1}{3}}}{c}$. *Ans.* $+\frac{a^{\frac{5}{2}}b^{\frac{5}{3}}}{c^5}$.

19. Find the 6^{th} power of $2a^{\frac{1}{3}}b^{\frac{1}{6}}$. *Ans.* $64a^2b$.

20. Find the 6^{th} power of $-2a^{\frac{1}{3}}b^{\frac{1}{6}}$. *Ans.* $64a^2b$.

21. Find the 3^d power of $2a^{\frac{1}{3}}b^{\frac{1}{6}}$. *Ans.* $8ab^{\frac{1}{2}}$.

22. Find the 3^d power of $-2a^{\frac{1}{3}}b^{\frac{1}{6}}$. *Ans.* $-8ab^{\frac{1}{2}}$.

23. Find the t^{th} power of $Ma^{\frac{1}{n}}b^{\frac{2}{p}}$. *Ans.* $(M)^t a^{\frac{t}{n}}b^{\frac{2t}{p}}$.

EXTRACTION OF ROOTS OF QUANTITIES AFFECTED WITH ENTIRE OR FRACTIONAL EXPONENTS.—MONOMIALS.

290. To extract the m^{th} root of any quantity, a^p, is to find a second quantity which, when raised to the m^{th} power, will produce the given quantity, a^p. The extraction of a root is then just the reverse of raising to a power, and, of course, the reverse process must be pursued to find the root. The m^{th} root of a^p must be $a^{\frac{p}{m}}$, because the m^{th} power of $a^{\frac{p}{m}}$ is equal to a^p. To raise a quantity, which has a coefficient, to a power, we first raise the coefficient to the indicated power, and write the new coefficient before the literal factors as the coefficient of the power. Hence, to extract the root of a quantity, which has a coefficient, we must first extract the root of the coefficient, and write it before the root of the literal factors. The m^{th} power of Ma^p is $(M)^m a^{pm}$, therefore, the m^{th} root of $(M)^m a^{pm}$ must be Ma^p. The m^{th} power of the fraction $\frac{a^r}{b^s} = \frac{a^{mr}}{b^{ms}}$. Hence, the m^{th} root of $\frac{a^{mr}}{b^{ms}} = \frac{a^r}{b^s}$. That is, the m^{th} root of numerator and denominator must be taken separately.

RULE.

Extract the root of the coefficient, and write the result before the literal factors affected with exponents equal to the quotient arising from dividing their primitive exponents by the index of the root. If the coefficient of the given monomial is not a perfect power of the degree of the root to be extracted, the operation is impossible. If the exponents of the literal factors are not divisible by the index of the root, the literal factors will appear in the root with fractional exponents. The root of fractions is extracted by extracting the root of the numerator and denominator separately.

EXAMPLES.

1. Find the $\frac{1}{r}^{th}$ root of $a^{\frac{m}{r}}$. *Ans.* a^m.

2. Find the r^{th} root of $a^{\frac{m}{r}}$. *Ans.* $a^{\frac{m}{r^2}}$.

3. Find the m^{th} root of $a^{\frac{m}{r}}$. *Ans.* $a^{\frac{1}{r}}$.

4. Find the $-\frac{1}{r}^{th}$ root of $a^{\frac{m}{r}}$. *Ans.* a^{-m}.

5. Find the $-m^{th}$ root of $a^{\frac{m}{r}}$. *Ans.* $a^{-\frac{1}{r}}$.

6. Find the $-r^{th}$ root of $a^{\frac{m}{r}}$. *Ans.* $a^{-\frac{m}{r^2}}$.

7. Find the $-m^{th}$ root of $a^{-\frac{m}{r}}$. *Ans.* $a^{\frac{1}{r}}$.

8. Find the $-\frac{1}{m}^{th}$ root of $a^{-\frac{m}{r}}$. *Ans.* $a^{\frac{m^2}{r}}$.

9. Find the 4^{th} root of $81a^8c^{12}b^{16}$. *Ans.* $3a^2c^3b^4$.

10. Find the 7^{th} root of $128a^7b^{-7}$. *Ans.* $2ab^{-1}$.

11. Find the q^{th} root of $(2)^q a^q b^{-q}$. *Ans.* $2ab^{-1}$.

12. Find the $-5m^{th}$ root of $a^{-10m}b^{-15m}$. *Ans.* a^2b^3.

13. Find the r^{th} root of a^{mnr}. *Ans.* a^{mn}.

14. Find the 4^{th} root of $16a^{\frac{1}{12}}b^{\frac{12}{8}}$. *Ans.* $2a^{\frac{1}{48}}b^{\frac{3}{8}}$.

15. Find the 5^{th} root of $\frac{a^{\frac{5}{2}}b^{\frac{5}{3}}}{c^5}$. *Ans.* $\frac{a^{\frac{1}{2}}b^{\frac{1}{3}}}{c}$.

16. Find the 5^{th} root of $-\frac{a^{\frac{5}{2}}b^{\frac{5}{3}}}{c^5}$. *Ans.* $-\frac{a^{\frac{1}{2}}b^{\frac{1}{3}}}{c}$.

17. Find the 3^d root of $8ab^{\frac{1}{2}}$. *Ans.* $2a^{\frac{1}{3}}b^{\frac{1}{6}}$.

18. Find the 6^{th} root of $64a^2b$. *Ans.* $2a^{\frac{1}{3}}b^{\frac{1}{6}}$.

19. Find the 3^d root of $-8ab^{\frac{1}{2}}$. *Ans.* $-2a^{\frac{1}{3}}b^{\frac{1}{6}}$.

20. Find the t^{th} root of $(M)^t a^{\frac{t}{n}} b^{\frac{2t}{p}}$. *Ans.* $Ma^{\frac{1}{n}}b^{\frac{2}{p}}$.

PROMISCUOUS EXAMPLES.

1. Raise $2a^{-\frac{1}{2}}b^{-\frac{1}{p}}c^{\frac{1}{3}}$ to the 3^d power. *Ans.* $8a^{-\frac{3}{2}}b^{-\frac{3}{p}}c$.

2. Extract the 3^d root of $8a^{-\frac{3}{2}}b^{-\frac{3}{p}}$. *Ans.* $2a^{-\frac{1}{2}}b^{-\frac{1}{p}}c^{\frac{1}{3}}$.

3. Multiply $2a^{-p}b^{-m}c^{-n}$ by $3a^{p}b^{m}c^{n}$. *Ans.* 6

4. Divide 6 by $3a^{p}b^{m}c^{n}$. *Ans.* $2a^{-p}b^{-m}c^{-n}$.

5. Divide 6 by $2a^{-p}b^{-m}c^{-n}$. *Ans.* $3a^{p}b^{m}c^{n}$.

6. Raise $2a^{\frac{1}{3}}b^{\frac{1}{6}}c^{\frac{1}{9}}$ to the 6^{th} power. *Ans.* $64a^{2}bc^{\frac{2}{3}}$.

7. Extract the 6^{th} root of $64a^{2}bc^{\frac{2}{3}}$. *Ans.* $2a^{\frac{1}{3}}b^{\frac{1}{6}}c^{\frac{1}{9}}$.

8. Multiply $(M)^{\frac{1}{m}}a^{\frac{1}{n}}$ by $(M)^{\frac{m-1}{m}}a^{\frac{n-1}{n}}$. *Ans.* Ma.

9. Divide Ma by $(M)^{\frac{m-1}{m}}a^{\frac{n-1}{n}}$. *Ans.* $(M)^{\frac{1}{m}}a^{\frac{1}{n}}$.

10. Divide Ma by $(M)^{\frac{1}{m}}a^{\frac{1}{n}}$. *Ans.* $(M)^{\frac{m-1}{m}}a^{\frac{n-1}{n}}$.

11. Multiply $8x^{-\frac{1}{3}}y^{-\frac{1}{4}}$ by $\frac{x^{\frac{4}{3}}y^{\frac{5}{4}}}{8}$. *Ans.* xy.

12. Divide xy by $8x^{-\frac{1}{3}}y^{-\frac{1}{4}}$. *Ans.* $\frac{x^{\frac{4}{3}}y^{\frac{5}{4}}}{8}$.

13. Divide xy by $\frac{x^{\frac{4}{3}}y^{\frac{5}{4}}}{8}$. *Ans.* $8x^{-\frac{1}{3}}y^{\frac{1}{4}}$.

14. Raise $-\frac{5x^{-1}y^{-2}z^{-3}}{ab}$ to the third power.

Ans. $-125x^{-3}y^{-6}z^{-9}a^{-3}b^{-3}$.

15. Raise $Pa^{2}b^{3}$ to the r^{th} power. *Ans.* $(P)^{r}a^{2r}b^{3r}$.

16. Extract the r^{th} root of $(P)^{r}a^{2r}b^{3r}$. *Ans.* $Pa^{2}b^{3}$.

17. Divide $(2)^{3}a^{3}b^{2}$ by $(2)^{2}a^{m}b^{m}$. *Ans.* $2a^{3-m}b^{2-m}$.

18. Multiply $2a^{3-m}b^{2-m}$ by $(2)^{2}a^{m}b^{m}$. *Ans.* $8a^{3}b^{2}$.

19. Multiply $2a^{3-m}b^{2-m}$ by $(2)^{3}a^{4m}b^{3m}$. *Ans.* $(2)^{4}a^{3(1+m)}b^{2(1+m)}$.

20. Extract the s^{th} root of $(W)^{2s}a^{-s}b^{-\frac{1}{s}}$. *Ans.* $(W)^{2}a^{-1}b^{-\frac{1}{s^2}}$.

21. Raise $3a^{\frac{1}{r}}b^{\frac{2}{r}}$ to the 10^{th} power. *Ans.* $(3)^{10}a^{\frac{10}{r}}b^{\frac{20}{r}}$.

22. Extract the 10^{th} root of $(3)^{10}a^{\frac{10}{r}}b^{\frac{20}{r}}$. *Ans.* $3a^{\frac{1}{r}}b^{\frac{2}{r}}$.

23. Multiply $(4)^{7}a^{\frac{1}{2}}b^{\frac{1}{6}}$ by $(4)^{2}a^{\frac{3}{2}}b^{\frac{5}{6}}$. *Ans.* $(4)^{9}a^{2}b$.

24. Divide $(4)^9a^2b$ by $(4)^2a^{\frac{3}{2}}b^{\frac{5}{6}}$. *Ans.* $(4)^7a^{\frac{1}{2}}b^{\frac{1}{6}}$.

25. Divide $(4)^9a^2b$ by $(4)^7a^{\frac{1}{2}}b^{\frac{1}{6}}$. *Ans.* $(4)^2a^{\frac{3}{2}}b^{\frac{5}{6}}$.

26. Raise $-a^{\frac{1}{2}}b^{\frac{1}{5}}$ to the 3rd power. *Ans.* $-a^{\frac{3}{2}}b^{\frac{3}{5}}$.

27. Raise $-a^{\frac{1}{2}}b^{\frac{1}{5}}$ to the 4th power. *Ans.* $+a^2b^{\frac{4}{5}}$.

28. Raise $2x^{\frac{1}{c}}y^{\frac{2}{c}}z^{\frac{3}{c}}$ to the cth power. *Ans.* $(2)^cxy^2z^3$.

29. Extract the cth root of $(2)^cxy^2z^3$. *Ans.* $2x^{\frac{1}{c}}y^{\frac{2}{c}}z^{\frac{3}{c}}$.

30. Find the rth root of $\frac{(2)}{a^m}$. *Ans.* $2a^{-\frac{m}{r}}$.

31. Raise $2a^{-\frac{m}{r}}$ to the rth power. *Ans.* $(2)^r a^{-m}$.

32. Raise $a^{.5}b^{.6}$ to the 10th power. *Ans.* a^5b^6.

33. Extract the 10th root of a^5b^6. *Ans.* $a^{.5}b^{.6}$.

34. Raise $a^{.5}b^{.6}$ to the 2d power. *Ans.* $ab^{1.2}$

35. Extract the square root of $ab^{1.2}$. *Ans.* $a^{.5}b^{.6}$.

36. Raise $a^{.002}b^{.003}$ to the 6th power. *Ans.* $a^{.012}b^{.018}$.

37. Extract the 6th root of $a^{.012}b^{.018}$. *Ans.* $a^{.002}b^{.003}$.

38. Raise $a^{.002}b^{.003}$ to the 1000th power. *Ans.* a^2b^3.

39. Extract the 1000th root of a^2b^3. *Ans.* $a^{.002}b^{.003}$.

40. Multiply $a^{\frac{1}{2}}b^{\frac{1}{2}}$ by $a^{.25}b^{.25}$. *Ans.* $a^{.75}b^{.75}$.

41. Divide $a^{.75}b^{.75}$ by $a^{\frac{1}{2}}b^{\frac{1}{2}}$. *Ans.* $a^{.25}b^{.25}$.

42. Divide $a^{.75}b^{.75}$ by $a^{.25}b^{.25}$. *Ans.* $a^{.5}b^{.5}$.

43. Multiply $(10)^2a^{.2}b^{.3}$ by $(10)^3a^{.3}b^{.2}$. *Ans.* $(10)^5a^{.5}b^{.5}$.

44. Divide $(10)^5a^{.5}b^{.5}$ by $(10)^3a^{.3}b^{.2}$. *Ans.* $(10)^2a^{.2}b^{.3}$.

45. Divide $(10)^5a^{.5}b^{.5}$ by $(10)^2a^{.2}b^{.3}$. *Ans.* $(10)^3a^{.3}b^{.2}$.

46. Multiply $(10)^{-1}a^{.4}b^{.6}$ by $(10)^3a^{-.2}b^{-.4}$. *Ans.* $(10)^2a^{.2}b^{.2}$.

47. Raise $2ab^2c^3$ to the $\frac{1}{10}$th power. *Ans.* $(2)^{\cdot 1}a^{\cdot 1}b^{\cdot 2}c^{\cdot 3}$.

48. Extract the $\frac{1}{10}$th root of $(2)^{\cdot 1}a^{\cdot 1}b^{\cdot 2}c^{\cdot 3}$. *Ans.* $2ab^2c^3$.

49. Raise $(2)^3xy^2z^3$ to the $\frac{3}{10}$th power. *Ans.* $(2)^{\cdot 9}x^{\cdot 3}y^{\cdot 6}z^{\cdot 9}$.

50. Raise $(2)^5a^mb^n$ to the $\frac{2}{10}$th power. *Ans.* $2a^{\cdot 2m}b^{\cdot 2n}$.

It will be seen that decimal powers are smaller, and decimal roots greater, than the quantities themselves.

CALCULUS OF RADICALS.

291. Any quantity with a radical sign is called a radical quantity, or simply a radical. Thus, $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{c^3}$, are radical quantities, or radicals.

The coefficient of a radical is the quantity prefixed to the radical sign, and it indicates the number of times, plus one, that the radical has been added to itself. Thus, $2\sqrt{a}$ and $m\sqrt{a}$, indicate that $\sqrt{a}$ has been added to itself once and $m-1$ times. When no coefficient is written, unity is understood to be the coefficient. Thus, $\sqrt{a} = 1\sqrt{a}$. When the indicated root can be exactly extracted, the radical is said to be commensurable or rational. Thus, $\sqrt{4}$ and $\sqrt[3]{8}$ are rational radicals. When the indicated root cannot be exactly extracted, the radical is said to be incommensurable or irrational. Thus, $\sqrt{2}$ and $\sqrt[3]{5}$ are irrational or incommensurable radicals.

A root has been defined to be a quantity, which, taken as a factor a certain number of times, will produce the given quantity. An *even* root is a quantity, which, taken as a factor an *even* number of times, will produce the given quantity. But no quantity taken an even number of times will produce a negative result. Hence the even root of a negative quantity is impossible. The indicated even roots of negative quantities are called *imaginary quantities.* Thus, $\sqrt{-2}$, $\sqrt[4]{-2}$, $\sqrt[6]{-2}$, are imaginary quantities. Roots that are not imaginary are called *real* roots.

The term rational is in contradistinction to irrational.

The term real is in contradistinction to imaginary.

A quantity may be real and not rational; but no quantity can be rational or irrational and not be real. Thus, $\sqrt{2}$ is real, but not rational; but $\sqrt{4}$ and $\sqrt{2}$ are both real.

292. It has been shown that all radicals may be changed into parenthetical expressions; the numerator of the exponent of the parenthesis denoting the power to which the quantity under the radical sign is raised, and the denominator of the exponent of the parenthesis denoting the index of the radical, or the degree of the root to be extracted. Thus, $\sqrt{a^3}$ may be changed into $a^{\frac{3}{2}}$, and $\sqrt[n]{a^m}$ may be changed into $a^{\frac{m}{n}}$.

293. A *simple radical* is one, which, when changed into an equivalent parenthetical expression, has unity for the numerator of the exponent of the parenthesis. Thus $\sqrt[3]{a}$, $\sqrt[4]{a}$, &c., are simple radicals. A *complex radical* is one which has the quantity under the sign raised to some power different from unity. Thus, $\sqrt{a^3}$, $\sqrt[n]{a^m}$, are complex radicals, the equivalent parenthetical expressions $(a)^{\frac{3}{2}}$, and $(a)^{\frac{m}{n}}$, having numerators different from unity.

294. Radicals are said to be *similar* when they have the same index and the same quantity under the sign. Thus, $2\sqrt{a}$ and $3\sqrt{a}$ are similar radicals. The $\sqrt{a}$ and $\sqrt[3]{a}$ are not similar, because, though the quantities under the radical signs are the same, the indices of the radicals are different. The $\sqrt{a}$ and $\sqrt{b}$ are dissimilar, because the quantities under the signs are different.

295. Similar powers of the same quantity can be added by adding their coefficients. Thus, $2a^2 + 3a^2 = 5a^2$; because, since the literal factors are the same, they may be represented by the same letter, x. Then, $2a^2 + 3a^2 = 2x + 3x = 5x$, and replacing x by its value a^2, we have the sum of the two quantities equal to $5a^2$. For a like reason, similar powers of the same quantity may be subtracted from each other, by taking the difference between their coefficients and uniting this difference as the coefficient of the common quantity. Thus, $5a^3 - 2a^3 = 3a^3$.

296. In the same way, similar radicals may be added or subtracted, $2\sqrt{a} + \sqrt{a} = 3\sqrt{a}$. For, make $\sqrt{a} = x$; then $2\sqrt{a} + \sqrt{a} = 2x + x = 3x = 3\sqrt{a}$. So, also, $2\sqrt{a} - \sqrt{a} = \sqrt{a}$.

297. The calculus of radicals shows how radicals may be operated upon algebraically—added, subtracted, multiplied, &c., &c.

These algebraic operations must be in accordance with certain principles, which both modify and facilitate them. It would require an extended treatise to embrace all the principles of the calculus of radicals. A few of the most important only are given in this work.

First Principle.

298. Any parenthetical expression, composed of several factors, may be decomposed into as many new parenthetical expressions as there are factors. Thus, $(abc)^{\frac{p}{q}} = (a)^{\frac{p}{q}} (b)^{\frac{p}{q}} (c)^{\frac{p}{q}}$.

For by the rules for raising a monomial affected with any exponent, fractional or entire, to a power, each factor has to be separately raised to the indicated power. Hence, $(abc)^{\frac{p}{q}} = (a)^{\frac{p}{q}} (b)^{\frac{p}{q}} (c)^{\frac{p}{q}}$. And, in like manner, $(a^m b^n c^r)^{\frac{p}{q}} = (a^m)^{\frac{p}{q}} (b^n)^{\frac{p}{q}} (c^r)^{\frac{p}{q}}$.

299. This principle has an important application in the case of simple radicals. For $(abcd)^{\frac{1}{n}} = \sqrt[n]{abcd}$, is also equal to $(a)^{\frac{1}{n}} (b)^{\frac{1}{n}} (c)^{\frac{1}{n}} (d)^{\frac{1}{n}}$ or to $\sqrt[n]{a} \,.\, \sqrt[n]{b} \,.\, \sqrt[n]{c} \, \sqrt[n]{d}$. Hence, $\sqrt[n]{abcd} = \sqrt[n]{a} \sqrt[n]{b} \sqrt[n]{c} \sqrt[n]{d}$. That is, *the* nth *root of the product of any number of factors is equal to the product of the* nth *roots of these factors.*

300. This property of radicals is used for two distinct purposes.

1st. To simplify radicals. Let it be required to simplify or reduce $\sqrt{75a^2c}$. We see that the quantity under the sign $75a^2c$, can be decomposed into two factors, $25a^2$ and $3c$, and that the first is a perfect power of the degree of the root to be extracted. Then, $\sqrt{75a^2c} = \sqrt{25a^2 \times 3c}$, which is also, by the property just demonstrated, $= \sqrt{25a^2}\ \sqrt{3c} = 5a\ \sqrt{3c}$. In like manner, $\sqrt[3]{8a^3c} = \sqrt[3]{8a^3} \times \sqrt[3]{c} = 2a\ \sqrt[3]{c}$. Then, to simplify or reduce a radical, we have the following

RULE.

Decompose the quantity under the sign into two factors, one of which shall be a perfect power of the degree of the root to be extracted. Extract the root of the perfect power, and write this root before the incommensurable factor.

301. 2d. The property, that the n^{th} root of the product of any number of factors is equal to the product of the n^{th} roots of those factors, is also used to make radicals similar which appear dissimilar. Then, after they have been made similar, they may be operated upon algebraically, added, subtracted, &c., like simple algebraic expressions.

Thus, $\sqrt{4a^2c}$ and $m\sqrt{c}$ appear to be dissimilar. But $\sqrt{4a^2c} = \sqrt{4a^2 \,.\, c} = \sqrt{4a^2}\ \sqrt{c} = 2a\sqrt{c}$. Hence, if it be required to add $\sqrt{4a^2c}$ and $m\ \sqrt{c}$, we may represent $\sqrt{c}$ by x, and the sum of the two quantities will be $2ax + mx$ or $(2a + m)x$. Now replace x by its value $\sqrt{c}$, and we have $\sqrt{4a^2c} + m\sqrt{c} = (2a + m)\sqrt{c}$. In like manner, $2b\sqrt[3]{8a^3c^5}$ and $-2ac\sqrt[3]{c^2b^3}$, appear to be dissimilar. But, $2b\sqrt[3]{8a^3c^5} = 2b\sqrt[3]{8a^3c^3 \times c^2} = 2b\sqrt[3]{8a^3c^3} \times \sqrt[3]{c^2} = 4acb\sqrt[3]{c^2}$, and $-2ac\sqrt[3]{c^2b^3} = -2ac\sqrt[3]{b^3(c^2)} = -2ac\sqrt[3]{b^3}\ \sqrt[3]{c^2} = -2acb\sqrt[3]{c^2}$. Hence, $2b\sqrt[3]{8a^3c^5} - 2ac\sqrt[3]{c^2b^3} = 2acb\sqrt[3]{c^2}$. To operate upon dissimilar radicals, we have then the following

RULE.

First make the radicals similar, then represent the common radical by x, *or conceive it to be so represented, and operate upon the new radical expressions according to the rules for simple algebraic expressions.*

The following examples will afford illustrations of the two applications of the principle, that $\sqrt[n]{abc}$, &c. $= \sqrt[n]{a} \, . \, \sqrt[n]{b}\ \sqrt[n]{c}$, &c. For the sake of simplicity, we will assume the own signs of all the radicals to be positive.

EXAMPLES.

1. Simplify the expression $\sqrt{16a^2c^3b^5}$. *Ans.* $4acb^2\sqrt{cb}$.

2. Simplify the expression $\sqrt[3]{27x^3y^6z^7}$. *Ans.* $3xy^2z^2\sqrt[3]{z}$.

3. Simplify the expression $\sqrt{50a^3b^5c^7}$. *Ans.* $5ab^2c\sqrt[3]{2abc}$.

4. Simplify the expression $\sqrt[3]{24a^5b^6c^9}$. *Ans.* $2ab^2c^3\sqrt[3]{3a^2}$.

5. Add the radicals $\sqrt{16a^3b^{10}c^2}$ and $+\ 2ab^5c\sqrt{a}$.
Ans. $6ab^5c\sqrt{a}$.

6. Add the radicals $\sqrt{50a^{2m}b^{4n}}$ and $+\ 5a^mb^{3n}\sqrt{2b^{-2n}}$.
Ans. $10a^mb^{2n}\sqrt{2}$.

7. Add the radicals $4\sqrt{a^{-r}b^{-s}c^{-p}}$ and $+\ 3a^{2r}b^{4s}c^{6p}\sqrt{a^{-5r}b^{-9s}c^{-13p}}$.
Ans. $7\sqrt{a^{-r}b^{-s}c^{-p}}$.

8. Add the radicals $4a^rb^2\ \sqrt{c^3}$ and $8c\ \sqrt{a^{2r}b^4c}$.
Ans. $12a^rb^2c\sqrt{c}$.

9. Reduce $\sqrt{27} + \sqrt{12} - \sqrt{75}$ to one sum.
Ans. $0\sqrt{3}$, or 0.

10. Reduce $\sqrt{50} + \sqrt{8} + \sqrt{32} + \sqrt{18} + \sqrt{162}$ to one sum.
Ans. $23\sqrt{2}$.

11. Reduce $2\sqrt[3]{c} + 4\sqrt[3]{c^4} + 2a\sqrt[3]{c^7}$ to one sum.
Ans. $(2 + 4c + 2ac^2)\sqrt[3]{c}$.

12. Reduce $a\sqrt{20} + b\sqrt{5} - c\sqrt{5b^2}$ to one sum.
Ans. $(2a + b - cb)\sqrt{5}$.

13. Reduce $ab\sqrt{72} + c\sqrt{8} + mn\sqrt{18c^2}$ to one sum.
Ans. $(6ab + 2c + 3mnc)\sqrt{2}$.

14. Reduce $5\sqrt[4]{8}$ and $\sqrt[4]{8a^4c^8}$ to one sum. *Ans.* $(5 + ac^2)\sqrt[4]{8}$.

15. Reduce $12\sqrt[6]{a^6b^{12}d}$ and $m\sqrt[6]{d^7}$. *Ans.* $(12ab^2 + md)\sqrt[6]{d}$.

16. Reduce $a\sqrt[r]{b^{3+r}c^{4+r}n}$ and $m\sqrt[r]{n^{r+1}b^3c^4}$.
Ans. $(abc + mn)\sqrt[r]{b^3c^4n}$.

17. Reduce $\sqrt{48} + 5a\sqrt{12} + 6b\sqrt{48} - 24b\sqrt{12}$.
Ans. $(2 + 5a - 12b)\sqrt{12}$.

18. Reduce $4a\sqrt{c^3} + 5ac^3\sqrt{c^{-3}} - 9ac\sqrt{c}$ to one sum. *Ans.* 0.

19. Reduce $12b\sqrt[m]{a^{m+}c^{2m}} + 5m\sqrt[m]{a}$ to one sum.
Ans. $(12bac^2 + 5m)\sqrt[m]{a}$.

20. Reduce $3\sqrt{a^3c^3b^3}$ and $-2abc\sqrt{abc}$ to one sum.
Ans. $abc\sqrt{abc}$.

The whole difficulty in solving these examples consists in finding the incommensurable factor common to all the radicals.

Second Principle.

302. The n^{th} root of the quotient of two quantities is equal to the quotient of their n^{th} roots. That is, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

For, to raise a fraction to any power, we raise the numerator and denominator, separately, to the required power. Hence, to extract any root, as the n^{th}, the root of each term of the fraction must be extracted separately.

This principle is used in extracting the roots of fractions.

Third Principle.

303. The mn^{th} root of any quantity is equal to the m^{th} root of the n^{th} root of that quantity. That is, $\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}$.

A quantity, a, may be raised to the sixth power by first squaring a, and then cubing the result. Hence, the sixth root of a^6 might be extracted by first extracting the square root, and then extracting the cube root of the result. So, a may be raised to the mn^{th} power by first raising it to the n^{th} power, and then raising the result to the m^{th} power. Hence, the mn^{th} root can plainly be extracted by taking first the n^{th}, and then the m^{th} root, in succession.

304. This principle is of great importance, and extensive application. It is used to extract high roots in succession, whenever their indices can be decomposed into factors.

Thus, $\sqrt[4]{256a^8b^{12}} = \sqrt{\sqrt{256a^8b^{12}}} = \sqrt{16a^4b^6} = 4a^2b^3$.

All even roots, and roots that are multiples of 3, can be extracted in succession. The sixth root is equal to the cube root of the square root. The eighth root is equal to the fourth root of the square root, or it can be found by extracting the square root three times. But the fifth root, seventh root, eleventh root, &c., cannot be extracted successively.

When the principle can be applied, we begin by extracting the lowest root first. The index of the radical must be decomposed into its prime factors, and the root corresponding to the lowest factor ought to be first extracted. It frequently happens that the factors are equal.

EXAMPLES.

1. Extract the eighth root of $256a^8b^8$.

$$\sqrt[8]{256a^8b^8} = \sqrt[4]{\sqrt{256a^8b^8}} = \sqrt[4]{16a^4b^4} = \sqrt{\sqrt{16a^4b^4}} = \sqrt{4a^2b^2} = 2ab.$$

2. Extract the sixth root of $729a^{18}b^{24}$.

$$\sqrt[6]{729a^{18}b^{24}} = \sqrt[3]{\sqrt{729a^{18}b^{24}}} = \sqrt[3]{27a^9b^{12}} = 3a^3b^4.$$

3. Required $\sqrt[4]{6561a^{24}b^{32}}$. *Ans.* $9a^6b^8$.

4. Required $\sqrt[4]{256a^2b^2}$. *Ans.* $4a^{\frac{1}{2}}b^{\frac{1}{2}}$.

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5. Required $\sqrt[6]{4096ab}$. *Ans.* $4a^{\frac{1}{6}}b^{\frac{1}{6}}$.

6. Required $\sqrt[8]{65536a^{\frac{4}{3}}b^{\frac{4}{3}}}$. *Ans.* $4a^{\frac{1}{6}}b^{\frac{1}{6}}$.

7. Required $\sqrt[8]{6561a^{8p}b^{8q}}$. *Ans.* $3a^pb^q$.

8. Required $\sqrt[2s]{a^{2s}b^{4s}}$. *Ans.* ab^2.

9. Required $\sqrt[3s]{(27)^sa^{6s}b^{12s}}$. *Ans.* $3a^2b^4$.

In this example, extract the s^{th} root first.

Fourth Principle.

305. The coefficient of a radical may be passed under the radical sign by raising it to a power indicated by the index of the radical. That is, $P\sqrt[m]{a} = \sqrt[m]{P^ma}$. For, $P = \sqrt[m]{P^m}$; hence, $P\sqrt[m]{a} = \sqrt[m]{P^m}\ \sqrt[m]{a}$; but, by the first principle, $\sqrt[m]{P^m}\ \sqrt[m]{a} = \sqrt[m]{P^ma}$. Therefore, $P\sqrt[m]{a} = \sqrt[m]{P^ma}$.

306. The fourth principle is, obviously, just the converse of the first principle. The latter enables us to pass a factor outside of the radical; the former, to place a factor under the radical. The fourth principle is frequently used in the differential calculus. It has two applications in Algebra: 1st. The approximate value of incommensurable roots can sometimes be found more exactly by means of this principle. Thus, $\frac{2}{3}\sqrt{7} = \sqrt{(\frac{2}{3})^2 \cdot 7} = \sqrt{\frac{28}{9}} = \frac{5}{3}$: true to within less than $\frac{1}{3}$. A nearer approximation to the value of the expression has been found in this instance by passing the coefficient under the radical. But, for such expressions as $\frac{2}{3}\sqrt{2}$, $\frac{2}{3}\sqrt{5}$, nothing is gained by passing the coefficient under the radical.

2d. The fourth principle may be used instead of the first, or in connection with the first, in making radicals similar which appear dissimilar. Thus, $9a\sqrt{b}$, and $\sqrt{81a^2b}$ are dissimilar, until either $9a$ is passed under the radical, or $81a^2$ is passed without. Sometimes it is difficult to employ the first principle to make radicals similar, because it is not easy to perceive the incommensurable factor common to all the radicals. But there is no difficulty in employing the fourth principle. Pass all the coefficients under their respective radical signs, and then decompose the numerical and literal factors into their prime factors. If the radicals can be made similar, common factors will become apparent after the decomposition into prime factors. Let it be required to ascertain whether $2a\sqrt[3]{ab^5}$, and $b\sqrt[3]{a^4b^2}$ are similar. The equivalent expressions are $\sqrt[3]{8a^4b^5}$, and $\sqrt[3]{a^4b^5}$. The two radicals have, then, a common factor, a^4b^5, under the sign. This common factor can be decomposed into two others, one of which is a perfect power of the degree of the root to be extracted, and the other incommensurable. The incommensurable factor may be made a new radical, and the given expressions being decomposed into factors, one of which is this incommensurable factor, will be similar. We have, in accordance with this rule, $2a\sqrt[3]{ab^5} = \sqrt[3]{8a^4b^5} = \sqrt[3]{8a^3b^3}\sqrt[3]{ab^2} = 2ab\sqrt[3]{ab^2}$, and $b\sqrt[3]{a^4b^2} = \sqrt[3]{a^4b^5} = \sqrt[3]{a^3b^3}\sqrt[3]{ab^2} = ab\sqrt[3]{ab^2}$.

Take, as a second example, $7\sqrt[4]{x^5y^3}$, and $x\sqrt{2401xy}$; by passing 7 and x under the radical signs, we have $\sqrt[4]{2401x^5y^3}$, and $\sqrt[4]{2401x^5y}$. Hence, $\sqrt[4]{2401x^5}$ is a common factor, and the given expressions reduce to $7x\sqrt[4]{xy^3}$ and $7x\sqrt[4]{xy}$, and are not similar, since they have no common incommensurable factor. We have, then, a simple process for ascertaining whether radical expressions can be made similar.

RULE.

Pass the coefficients of the different radicals under their respective signs, and then examine whether there is a common incommensurable factor. If so, the radicals may be made similar by taking the incommensurable factor as a new radical. Write before it, for new coefficients, the roots of the commensurable factors of the respective expressions, and we will have a series of new radical expressions, all similar. If there is no common incommensurable factor, the radical expressions are not similar.

EXAMPLES.

1. Add together $2a\sqrt{bc^3}$ and $3cb^2\sqrt{bc^2}$.

Then, $2a\sqrt{bc^3} = \sqrt{4a^2bc^3} = \sqrt{4a^2c^2}\sqrt{bc}$ and $3cb^2\sqrt{bc^2} = \sqrt{9b^3c^3} = \sqrt{9b^2c^2}\sqrt{bc}$. Hence, $\sqrt{bc}$ is a common incommensurable factor, and the sum is $(2ac + 3bc)\sqrt{bc}$.

2. Add $4x^{\frac{1}{2}}y^3\sqrt{a^3c^3}$ and $acz\sqrt{ab^4c}$ together.

Ans. $(4x^{\frac{1}{2}}y^3 + b^2z)\,ac\sqrt{ac}$.

3. Add together $b\sqrt{540a^3}$ and $6a\sqrt{240a}$.

$b\sqrt{540a^3} = \sqrt{3.5.2^2.3^2a^3b^2}$ and $6a\sqrt{240a} = \sqrt{36.3.5.4^2a^3}$. Hence, $\sqrt{3.5.a}$, or $\sqrt{15a}$, is a common incommensurable factor, and the sum of the two expressions is $(6ab + 24a)\sqrt{15a}$.

4. Add together $\sqrt{90}$ and $\sqrt{810}$.

$\sqrt{90} = \sqrt{3^2.5.2}$ and $\sqrt{810} = \sqrt{3^4.2.5}$. Hence, $\sqrt{2.5}$ is the common incommensurable factor, and the sum is $(3 + 9)\sqrt{10}$.

5. Add together $a\sqrt{216}$ and $b\sqrt{2400}$. *Ans.* $(6a + 20b)\sqrt{6}$.

6. From $a\sqrt{300}$ subtract $b\sqrt{243}$. *Ans.* $(10a - 9b)\sqrt{3}$.

7. From $a\sqrt[3]{33750}$ subtract $b\sqrt[3]{10000}$.
Ans. $(3a - 2b)\sqrt[3]{1250}$.

8. Add the three expressions, $ba\sqrt{ax}$, $mx^{\frac{1}{2}}\sqrt{9a^3}$, $na^{\frac{1}{2}}\sqrt{x}$.
Ans. $(ba + 3ma + n)\sqrt{ax}$.

9. Add the three expressions, $ba\sqrt[3]{ax}$, $mx^{\frac{1}{3}}\sqrt[3]{27a^4} + na^{\frac{1}{3}}\sqrt[3]{x}$.
Ans. $(ba + 3ma + n)\sqrt[3]{ax}$.

10. Add the three expressions, $ba\sqrt[4]{ax} + mx^{\frac{1}{4}}\sqrt[4]{81a^5} + na^{\frac{1}{4}}\sqrt[4]{x}$.
Ans. $(ba + 3ma + n)\sqrt[4]{ax}$.

Fifth Principle.

307. Any factor may be passed without a parenthetical expression, by multiplying its exponent by the exponent of the parenthesis.

Thus, $(a^mbc)^{\frac{p}{q}} = a^{\frac{mp}{q}}(bc)^{\frac{p}{q}}$.

For, $(a^mbc)^{\frac{p}{q}} = a^{\frac{mp}{q}}b^{\frac{p}{q}}c^{\frac{p}{q}}$, since each term has to be raised to the $\frac{p}{q}$ power. And, by separating into factors, we have $a^{\frac{mp}{q}} \times b^{\frac{p}{q}}c^{\frac{p}{q}}$, which is also equal to $a^{\frac{mp}{q}}(bc)^{\frac{p}{q}}$.

If there is more than one term within the parenthesis, any term or a factor of any term may be passed out, by dividing all the terms by the expression to be passed out, and then writing that expression outside of the parenthesis, with its primitive exponent multiplied by the exponent of the parenthesis. Let it be required to pass a^m without the parenthesis $(a^m b + c)^{\frac{p}{q}}$. This may be written $\left|a^m\left(b + \frac{c}{a^m}\right)\right|^{\frac{p}{q}} = (a^m d)^{\frac{p}{q}}$ $= a^{\frac{mp}{q}}(d)^{\frac{p}{q}}$. In which $d = b + \frac{c}{a^m}$ or $b + a^{-m}c$. Hence, $(a^m b + c)^{\frac{p}{q}}$ $= a^{\frac{mp}{q}}(b + a^{-m}c)^{\frac{p}{q}}$.

It matters not how many terms there may be within the parenthesis; they may be all represented by a single letter, after they have been divided by the factor to be passed without. The foregoing demonstration will then be applicable.

EXAMPLES.

1. Pass x without the parenthesis $(xy^2z^3)^{-\frac{1}{5}}$.
   Ans. $x^{-\frac{1}{5}}(y^2z^3)^{-\frac{1}{5}}$.

2. Pass x^{-3} without the parenthesis $(x^{-3}y^2z^3)^{\frac{m}{n}}$.
   Ans. $x^{-\frac{3m}{n}}(y^2z^3)^{\frac{m}{n}}$.

3. Pass $x^{-\frac{1}{3}}$ without the parenthesis $(x^{-\frac{1}{3}}ab)^{-3}$.
   Ans. $x(ab)^{-3}$.

4. Pass 4 without the parenthesis $(4a^2b^3)^{\frac{1}{2}}$. *Ans.* $2(a^2b^3)^{\frac{1}{2}}$.

5. Pass $x^{\frac{p}{q}}$ without the parenthesis $(x^{\frac{p}{q}}y^m z^n)^{\frac{q}{p}}$. *Ans.* $x(y^m z^n)^{\frac{q}{p}}$.

6. Pass 8 without the parenthesis $(8ab)^{\frac{1}{3}}$. *Ans.* $2(ab)^{\frac{1}{3}}$.

7. Pass 16 without the parenthesis $(16ab)^{\frac{1}{4}}$. *Ans.* $2(ab)^{\frac{1}{4}}$.

8. Pass 4 without the parenthesis $\frac{(4a + b)^{\frac{3}{2}}}{8}$. *Ans.* $\left(a + \frac{b}{4}\right)^{\frac{3}{2}}$.

9. Pass 5 without the parenthesis $(a + 5b)^3$.
   Ans. $125\left(\frac{a}{5} + b\right)^3$.

10. Pass 2 without the parenthesis $(2a + 4b)^4$.
    Ans. $16(a + 2b)^4$.

11. Pass 4 without the parenthesis $\frac{(4\,(mx + nx^2) + (m + 2nx)^2\,)^{\frac{3}{2}}}{2m^2}$.

$$Ans.\quad \frac{(mx + nx^2 + \frac{1}{4}\,(m + 2nx)^2\,)^{\frac{3}{2}}}{\frac{m^2}{4}}.$$

12. Pass the term $4a$ without the parenthesis $(4a + b + c)^{\frac{1}{2}}$.

$$Ans.\quad 2a^{\frac{1}{2}}\left(1 + \frac{a^{-1}(b + c)}{4}\right)^{\frac{1}{2}}.$$

13 Pass the term $8a^3$ without the parenthesis $(8a^3 + b + c)^{\frac{1}{3}}$.

$$Ans.\quad 2a\left(1 + \frac{a^{-3}(b + c)}{8}\right)^{\frac{1}{3}}.$$

14. Pass the term $5a^2$ without the parenthesis $(5a^2 + b + c)^{\frac{1}{3}}$.

$$Ans.\quad (5)^{\frac{1}{3}}a^{\frac{2}{3}}\left(1 + \frac{a^{-2}(b + c)}{5}\right)^{\frac{1}{3}}.$$

Any number of terms may be passed without, by representing them by a single letter, and then replacing that letter by its value after the transfer has been made.

15. Pass $(a + b)$ without the parenthesis $(a + b + c)^{\frac{1}{2}}$. Let $a + b = x$. Then, $(a + b + c)^{\frac{1}{2}} = (x + c)^{\frac{1}{2}} = x^{\frac{1}{2}}(1 + x^{-1}c)^{\frac{1}{2}} = (a + b)^{\frac{1}{2}}\,|\,1 + (a + b)^{-1}c\,|^{\frac{1}{2}}$.

16. Pass $(a + b + c)$ without the parenthesis $(a + b + c + d)^{\frac{1}{2}}$.

$$Ans.\quad (a + b + c)^{\frac{1}{2}}\,|\,1 + (a + b + c)^{-1}d\,|^{\frac{1}{2}}.$$

17. Pass $(a + b + c)$ without the parenthesis $(a + b + c)^2$.

$$Ans.\quad (a + b + c)^2\,(1)^2 \text{ or } (a + b + c)^2.$$

18. Pass a^{-n} without the parenthesis $(a^{-n} + b)^{\frac{1}{p}}$.

$$Ans.\quad a^{-\frac{n}{p}}(1 + a^n b)^{\frac{1}{p}}.$$

The last example shows that any factor affected with a negative exponent may be made to appear with a positive exponent in the other terms of the parenthesis. This transformation is used in the differential calculus when it is desired to change a negative into a positive exponent.

It is evident that the fifth principle is only the more extended application of the first principle.

Sixth Principle.

308. Any factor may be passed within a parenthesis, by multiplying its exponent by the reciprocal of the parenthesis.

Because, to pass it out again, we must multiply its new exponent by the exponent of the parenthesis, and when its new exponent has been formed as directed, the factor, after it has been passed out again, will be affected with its primitive exponent. Thus, $a^m(b)^{\frac{p}{q}} = (a^{\frac{mq}{p}}b)^{\frac{p}{q}}$. Because, when $a^{\frac{mq}{p}}$ is passed out again, the expression will become $a^{\frac{mq}{p}\times\frac{p}{q}}(b)^{\frac{p}{q}} = a^m(b)^{\frac{p}{q}}$.

EXAMPLES.

1. Pass the coefficient 2 within the parenthesis $(a+b)^{\frac{1}{2}}$.

Ans. $|\,(2)^2a + (2)^2b\,|^{\frac{1}{2}}$.

2. Pass the coefficient 2 within the parenthesis $(a+b)^{\frac{1}{3}}$.

Ans. $(8a + 8b)^{\frac{1}{3}}$.

3. Pass a^m within the parenthesis $(a^{\frac{q}{p}} + b)^{\frac{p}{q}}$.

Ans. $(a^{\frac{mq}{p}+\frac{q}{p}} + a^{\frac{mq}{p}}b)^{\frac{p}{q}}$.

4. Pass a^m within the parenthesis $(a^{-\frac{mq}{p}} + b)^{\frac{p}{q}}$.

Ans. $(1 + a^{\frac{mq}{p}}b)^{\frac{p}{q}}$.

5. Pass a^m within the parenthesis $(a^m + b)^{-\frac{1}{3}}$.

Ans. $(a^{-2m} + ba^{-3m})^{-\frac{1}{3}}$.

6. Clear $8(\frac{1}{8}a + \frac{1}{4}b)^{\frac{3}{2}}$ of its coefficient. *Ans.* $\left(\frac{a}{2} + b\right)^{\frac{3}{2}}$.

7. Clear $64\left(\frac{32}{4096a} + \frac{32c}{16}\right)^{\frac{3}{2}}$ of its coefficient.

Ans. $\left(\frac{1}{8a} + 32c\right)^{\frac{3}{2}}$.

8. Clear $27(2a + \frac{1}{9}b)^{\frac{3}{2}}$ of its coefficient. *Ans.* $(18a + b)^{\frac{3}{2}}$.

9. Clear $32(2a + \frac{1}{4}b)^{\frac{5}{2}}$ of its coefficient. *Ans.* $(8a + b)^{\frac{5}{2}}$.

10 Clear the parenthetical expression $125\left(\frac{1}{25}a + \frac{1}{5}b\right)^{\frac{3}{4}}$ of its coefficient.

Ans. $(25a + 125b)^{\frac{3}{4}}$.

11. Clear the parenthetical expression $a^{\frac{s}{r}}(1 + a^{-1})^{\frac{r}{s}}$ of its coefficient.

Ans. $(a + 1)^{\frac{r}{s}}$.

It is evident that the sixth principle is only the more extended application of the fourth principle.

Seventh Principle.

309. The denominator of the exponent of a parenthetical expression may be multiplied by any quantity, provided we raise the quantity within the parenthesis to a power denoted by the multiplier. Thus, $(a)^{\frac{p}{q}} = (a^m)^{\frac{p}{mq}}$. For $(a)^{\frac{p}{q}} = a^{\frac{p}{q}}$, and $(a^m)^{\frac{p}{mq}} = a^{\frac{mp}{mq}} = a^{\frac{p}{q}}$. Hence, $(a)^{\frac{p}{q}} = (a^m)^{\frac{p}{mq}}$. If there is more than one term within the parenthesis, their algebraic sum may be represented by a single letter, and the foregoing demonstration is, therefore, applicable to all kinds of parenthetical expressions.

310. The seventh principle has two applications. 1st. It is used to cause complex radicals (or parenthetical expressions with the numerators of their exponents different from unity), to be affected with exponents having a common denominator.

Thus, $(a + b)^{\frac{3}{2}}$ and $(a + b)^{\frac{2}{3}}$, can be changed in accordance with the principle into the equivalent expressions $((a + b)^3)^{\frac{3}{6}}$ and $((a + b)^2)^{\frac{4}{6}}$.

We see that a second parenthesis has been written within each parenthesis; and we see, also, that the exponent of the first new parenthesis is the quotient arising from the division of 6, the least common multiple of the denominators of the exponents of the given parentheses, by the denominator of the exponent of the first given parenthesis. The exponent of the second new parenthesis is the quotient arising from dividing the same least common multiple, 6, by the denominator of the exponent of the second given parenthesis. In like manner,

$(a+b)^{\frac{2}{3}}$ and $(a+b)^{\frac{3}{4}}$ may be changed into $((a+b)^4)^{\frac{3}{12}}$ and $((a+b)^3)^{\frac{3}{12}}$.

Hence, to cause parenthetical expressions to be affected with exponents having a common denominator, we have the following

RULE.

Take the least common multiple of all the denominators of the exponents of the parentheses, and divide that multiple by the denominator of the exponent of every parenthesis. The several quotients will indicate the power to which the quantity within their respective parentheses must be raised.

EXAMPLES.

1. Change $(a+b)^{\frac{p}{m}}$ and $(a+b)^{\frac{p}{mn}}$, into equivalent parenthetical expressions, the denominators of whose exponents shall be the same.

Ans. $((a+b)^n)^{\frac{p}{mn}}$ and $(a+b)^{\frac{p}{mn}}$.

2. Change $(a+b)^{\frac{p}{m^2}}$ and $(a+b)^{\frac{p}{mn}}$, into equivalent expressions.

Ans. $((a+b)^n)^{\frac{p}{m^2n}}$ and $((a+b)^m)^{\frac{p}{m2n}}$.

311. 2d. But the most important application of the seventh principle, is in reducing simple radicals to a common index.

Thus, since $a^{\frac{1}{2}} = a^{\frac{3}{6}}$, or $\sqrt{a} = \sqrt[6]{a^3}$, and $a^{\frac{1}{3}} = a^{\frac{2}{6}}$, or $\sqrt[3]{a} = \sqrt[6]{a^2}$, it is plain that $\sqrt{a}$ and $\sqrt[3]{a}$, can be reduced to a common index. So, $\sqrt[n]{a}$ and $\sqrt[m]{a}$, may be changed into the equivalent radicals $\sqrt[mn]{a^m}$ and $\sqrt[mn]{a^n}$. In each instance, the power to which the quantity under the radical is raised is indicated by the quotient arising from dividing the least common multiple of the indices of all the radicals by the index of the radical under consideration.

RULE.

Form the least common multiple of the indices of all the radicals. Raise the quantity under each radical to a power indicated by the quotient arising from dividing the least common multiple by the index of the radical under consideration.

Reduce $\sqrt[3]{a}$, $\sqrt[4]{a}$, and $\sqrt[6]{a}$, to same index. The least common multiple is 12, and the three quotients $\frac{12}{3} = 4$, $\frac{12}{4} = 3$, and $\frac{12}{6} = 2$. Then the equivalent radicals are $\sqrt[12]{a^4}$, $\sqrt[12]{a^3}$, and $\sqrt[12]{a^2}$.

EXAMPLES.

1. Reduce $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[6]{a}$, and $\sqrt[8]{a}$, to same index.

Ans. $\sqrt[24]{a^8}$, $\sqrt[24]{a^6}$, $\sqrt[24]{a^4}$, and $\sqrt[24]{a^3}$.

2. Reduce $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[6]{a}$, $\sqrt[10]{a}$, and $\sqrt[30]{a}$, to same index.

Ans. $\sqrt[60]{a^{15}}$, $\sqrt[60]{a^{12}}$, $\sqrt[60]{a^{10}}$, $\sqrt[60]{a^6}$, and $\sqrt[60]{a^2}$.

3: Reduce $\sqrt[p]{a}$, $\sqrt[mp]{a}$, $\sqrt[np]{a}$, $\sqrt[m]{a}$, and $\sqrt[n]{a}$, to same index.

Ans. $\sqrt[mnp]{a^{mn}}$, $\sqrt[mnp]{a^n}$, $\sqrt[mnp]{a^m}$, $\sqrt[mnp]{a^{np}}$, and $\sqrt[mnp]{a^{mp}}$.

4. Reduce $\sqrt[s]{a}$, $\sqrt[s^2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, and $\sqrt[6]{a}$, to same index.

Ans. $\sqrt[12s^2]{a^{12s}}$, $\sqrt[12s^2]{a^{12}}$, $\sqrt[12s^2]{a^{4s^2}}$, $\sqrt[12s^2]{a^{3s^2}}$, and $\sqrt[12s^2]{a^{2s^2}}$.

5. Reduce $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[20]{a}$, $\sqrt[6]{a}$, $\sqrt[15]{a}$, $\sqrt[30]{a}$, and $\sqrt[60]{a}$, to same index.

Ans. $\sqrt[60]{a^{15}}$, $\sqrt[60]{a^{12}}$, $\sqrt[60]{a^3}$, $\sqrt[60]{a^{10}}$, $\sqrt[60]{a^4}$, $\sqrt[60]{a^2}$, and $\sqrt[60]{a}$.

6. Reduce $\sqrt[m]{a}$, $\sqrt[ms]{a}$, $\sqrt[ns]{a}$, $\sqrt[n]{a}$, $\sqrt[s]{a}$, and $\sqrt[mn]{a}$, to same index.

Ans. $\sqrt[mns]{a^{ns}}$, $\sqrt[mns]{a^n}$, $\sqrt[mns]{a^m}$, $\sqrt[mns]{a^{ms}}$, $\sqrt[mns]{a^{mn}}$, and $\sqrt[mns]{a^s}$.

7. Reduce $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[6]{a}$, $\sqrt[7]{a}$, $\sqrt[8]{a}$, and $\sqrt[9]{a}$, to same index.

Ans. $\sqrt[2520]{a^{1260}}$, $\sqrt[2520]{a^{840}}$, $\sqrt[2520]{a^{630}}$, $\sqrt[2520]{a^{504}}$, $\sqrt[2520]{a^{420}}$, $\sqrt[2520]{a^{360}}$, $\sqrt[2520]{a^{315}}$, and $\sqrt[2520]{a^{280}}$.

It will be seen that simple radicals are changed into complex by the operation of reduction to a common index.

Eighth Principle.

312. Any factor of the index of a complex radical may be suppressed, provided, the same factor is suppressed in the exponent of the power to which the quantity under the sign is raised. That is, $\sqrt[mn]{a^m} = \sqrt[n]{a}$.

For, $\sqrt[mn]{a^m} = (a)^{\frac{m}{mn}} = (a)^{\frac{1}{n}} = \sqrt[n]{a}$.

This principle is just the converse of the last, and reverses the results of the last. It is used to simplify complex radicals, and frequently reduces them to simple radicals.

EXAMPLES.

1. Simplify $\sqrt[6]{(a+b)^3}$, $\sqrt[6]{(a+b)^2}$, $\sqrt[12]{(a+b)^3}$ and $\sqrt[12]{(a+b)^2}$.
Ans. $\sqrt{a+b}$, $\sqrt[3]{a+b}$, $\sqrt[4]{a+b}$, and $\sqrt[6]{a+b}$.

2. Simplify $\sqrt[6]{(a+b)^9}$, $\sqrt[5]{(a+b)^{10}}$, $\sqrt[10]{(a+b)^5}$, and $\sqrt[8]{(a+b)^6}$.
Ans. $\sqrt{(a+b)^3}$, $(a+b)^2$, $\sqrt{a+b}$, and $\sqrt[4]{(a+b)^3}$.

3. Simplify $\sqrt{(a+b)^6}$, $\sqrt[6]{(a+b)^2}$, $\sqrt[12]{(a+b)^{18}}$, $\sqrt[5]{(a+b)^5}$, $\sqrt[9]{(a+b)^{18}}$.
Ans. $(a+b)^3$, $\sqrt[3]{(a+b)}$, $\sqrt{(a+b)^3}$, $(a+b)$, $(a+b)^2$

4. Reduce $\sqrt[24]{a^8}$, $\sqrt[24]{a^6}$, $\sqrt[24]{a^4}$, and $\sqrt[24]{a^3}$.
Ans. $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[6]{a}$, and $\sqrt[8]{a}$.

5. Reduce $\sqrt[60]{a^{15}}$, $\sqrt[60]{a^{12}}$, $\sqrt[60]{a^{10}}$, $\sqrt[60]{a^6}$, and $\sqrt[60]{a^2}$.
Ans. $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[6]{a}$, $\sqrt[10]{a}$, and $\sqrt[30]{a}$.

6. Reduce $\sqrt[mnp]{a^{mn}}$, $\sqrt[mnp]{a^n}$, $\sqrt[mnp]{a^m}$, $\sqrt[mnp]{a^{np}}$, $\sqrt[mnp]{a^{mp}}$.
Ans. $\sqrt[p]{a}$, $\sqrt[mp]{a}$, $\sqrt[np]{a}$, $\sqrt[m]{a}$, and $\sqrt[n]{a}$.

7. Reduce $\sqrt[60]{a^{15}}$, $\sqrt[60]{a^3}$, $\sqrt[60]{a^{10}}$, $\sqrt[60]{a^4}$, $\sqrt[60]{a^2}$, and $\sqrt[60]{a}$.
Ans. $\sqrt[4]{a}$, $\sqrt[20]{a}$, $\sqrt[6]{a}$, $\sqrt[15]{a}$, $\sqrt[30]{a}$, and $\sqrt[60]{a}$.

8. Reduce $\sqrt[mns]{a^{ns}}$, $\sqrt[mns]{a^n}$, $\sqrt[mns]{a^m}$, $\sqrt[mns]{a^{ms}}$, $\sqrt[mns]{a^{mn}}$, and $\sqrt[mns]{a^s}$.
Ans. $\sqrt[m]{a}$, $\sqrt[ms]{a}$, $\sqrt[ns]{a}$, $\sqrt[n]{a}$, $\sqrt[s]{a}$, and $\sqrt[mn]{a}$.

9. Reduce $\sqrt[2520]{a^{1260}}$, $\sqrt[2520]{a^{840}}$, $\sqrt[2520]{a^{630}}$, $\sqrt[2520]{a^{504}}$, $\sqrt[2520]{a^{420}}$, $\sqrt[2520]{a^{360}}$, $\sqrt[2520]{a^{315}}$, and $\sqrt[2520]{a^{280}}$.
Ans. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[6]{a}$, $\sqrt[7]{a}$, $\sqrt[8]{a}$, and $\sqrt[9]{a}$.

313. There are two consequences of the last two principles, of considerable importance. 1st. Whenever it is required to extract a root of a complex radical, which is a multiple of the exponent of the quantity under the sign, the extraction can be indicated by suppressing the exponent under the sign, and multiplying the index of the radical by the quotient, arising from dividing the index of the required root by

the exponent under the sign. Thus, let it be required to take the sixth root of $\sqrt{a^3}$, the result will be $\sqrt[4]{a}$. For, the sixth root of $\sqrt{a^3}$ is equal to $(a^{\frac{3}{2}})^{\frac{1}{6}} = a^{\frac{3}{12}} = a^{\frac{1}{4}} = \sqrt[4]{a}$. In like manner, the pn^{th} root of $\sqrt[m]{a^n} = \sqrt[pm]{a}$. For the pn^{th} root of $\sqrt[m]{a^n} = (a^{\frac{n}{m}})^{\frac{1}{pn}} = a^{\frac{n}{pnm}} = a^{\frac{1}{pm}} = \sqrt[pm]{a}$. These results have plainly been formed in accordance with the rule.

314. 2d. Whenever it is required to raise a simple radical to a power which is a factor of the index of the radical, the operation can be performed by dividing the index of the radical by the exponent of the power, and writing the quotient as the index of the radical instead of the old index. Thus, let it be required to raise $\sqrt[6]{a}$ to the square power. The result will be $\sqrt[3]{a}$; for the second power of the sixth root of a is $(a^{\frac{1}{6}})^2 = a^{\frac{2}{6}} = a^{\frac{1}{3}} = \sqrt[3]{a}$. So, likewise, the n^{th} power of $\sqrt[pn]{a} = \sqrt[p]{a}$. For the n^{th} power of $\sqrt[pn]{.\,a} = (a^{\frac{1}{pn}})^n = a^{\frac{n}{pn}} = a^{\frac{1}{p}} = \sqrt[p]{a}$.

The following are applications of the consequences.

EXAMPLES.

1. Required 4^{th} root of $\sqrt{a^8}$. *Ans.* $\sqrt[1]{a}$, or a.

2. Required 6^{th} root of $\sqrt{a^3}$. *Ans.* $\sqrt[4]{a}$.

3. Required 7^{th} root of $\sqrt{a^{14}}$. *Ans.* a.

4. Required 4^{th} root of $\sqrt[3]{a^2}$. *Ans.* $\sqrt[6]{a}$.

5. Required pm^{th} root of $\sqrt[n]{a^m}$. *Ans.* $\sqrt[pn]{a}$.

6. Required 12^{th} root of $\sqrt[5]{a^6}$. *Ans.* $\sqrt[10]{a}$.

7. Required 4^{th} power of $\sqrt[4]{a}$. *Ans.* a.

8. Required m^{th} power of $\sqrt[m]{a}$. *Ans.* a.

9. Required mn^{th} power of $\sqrt[m]{a}$. *Ans.* $\sqrt[\frac{1}{n}]{a}$, or a^n.

10. Required m^{th} power of $\sqrt[mn]{a}$. *Ans.* $\sqrt[n]{a}$.

11. Required 3^d power of $\sqrt[12]{a}$. *Ans.* $\sqrt[4]{a}$.

12. Required 12^{th} power of $\sqrt[3]{a}$. *Ans.* $\sqrt[\frac{1}{4}]{a}$, or a^4.

13. Required 6^{th} power of $\sqrt[30]{a}$. *Ans.* $\sqrt[5]{a}$.

14. Required 5^{th} power of $\sqrt[15]{a}$. *Ans.* $\sqrt[3]{a}$.

15. Required pm^{th} power of $\sqrt[pmn]{a}$. *Ans.* $\sqrt[n]{a}$.

TO MAKE SURDS RATIONAL BY MULTIPLICATION.

CASE I.

Monomial Surds.

315. Suppose the given surd is $\sqrt[n]{b}$; this is equivalent to $b^{\frac{1}{n}}$. Now, it is required to multiply $b^{\frac{1}{n}}$ by such a quantity as will make it rational. Since the surd will be rational when the numerator of the fractional exponent is exactly divisible by the denominator; and, since, in multiplication, we add the exponents of the same literal factors, the multiplier of $b^{\frac{1}{n}}$ must be b, affected with such an exponent, that, when added to $\frac{1}{n}$, the sum of the two exponents will be a whole number. Call x the unknown exponent of the multiplier, then $x+\frac{1}{n}=1$, or $x=1-\frac{1}{n}$ $=\frac{n-1}{n}$. Hence, the multiplier is $b^{\frac{n-1}{n}}$, and we see that $b^{\frac{1}{n}}\times b^{\frac{n-1}{n}}$ $=b$, a rational product. Had we placed $x+\frac{1}{n}=2$, and found the multiplier under this hypothesis, the product would have been b^2. But, when it is required to make the surd rational, and of the first degree, the sum of the primitive exponent, and the unknown exponent, must be placed equal to unity.

Let it be required to find a multiplier which will make $y^{\frac{3}{4}}$ rational. Then, $x+\frac{3}{4}=1$, or $x=\frac{1}{4}$. Hence, the multiplier is $y^{\frac{1}{4}}$, and we see that $y^{\frac{3}{4}}\cdot y^{\frac{1}{4}}=y$ is a rational product.

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RULE.

Place the primitive exponent, plus x, *equal to unity; find the value of* x *from this equation. The value so found will be the exponent of the multiplier. The multiplier itself must be the given monomial, exclusive of its exponent, raised to a power indicated by the value of* x.

EXAMPLES.

1. Find a multiplier which will make $x^{-\frac{m}{n}}$ rational.

Ans. $x^{\frac{m+n}{n}}$.

2. Find a multiplier that will make $y^{\frac{2}{5}}$ rational. *Ans.* $y^{\frac{3}{5}}$.

3. Find a multiplier that will make $y^{\frac{r+1}{n}}$ rational. *Ans.* $y^{\frac{n-(r+1)}{n}}$.

4. Find a multiplier that will make $y^{\frac{r+1}{n+m}}$ rational.

Ans. $y^{\frac{n+m-(r+1)}{n+m}}$.

Corollary.

316. The principles demontrated in Case I. enable us to find the approximate value of fractions whose denominators are monomial surds.

RULE.

Multiply both terms of the fraction by such a quantity as will make the denominator rational. Approximate as near as may be desired to the true value of the monomial surd in the numerator, and then reduce the fraction to its lowest terms.

EXAMPLES.

1. Required the approximate value of $\frac{5}{\sqrt{10}}$ to within ·01.

Ans. 1·58.

For, $\frac{5}{\sqrt{10}} = \frac{5\sqrt{10}}{10} = \frac{5(3{\cdot}16)}{10} = \frac{15{\cdot}80}{10} = 1{\cdot}58.$

2. Required the approximate value of $\frac{6}{\sqrt{12}}$ to within ·001.

Ans. 1·732.

3. Required the approximate value of $\frac{4}{\sqrt[3]{4}}$ to within ·01.

Ans. 2·52.

4. Required the approximate value of $\frac{100}{\sqrt{1000}}$ to within ·001.

Ans. 3·162.

5. Required the approximate value of $\frac{10}{\sqrt[3]{10}}$ to within ·01

Ans. 4·65.

6. Required the approximate value of $\frac{6}{\sqrt{1000}}$ to within ·0001.

Ans. ·189738, nearly.

7. Required the approximate value of $\frac{4000}{\sqrt[3]{400}}$ to within ·1.

Ans. 54·2.

8. Required the approximate value of $\frac{10}{\sqrt[3]{80}}$ to within ·01.

Ans. 2·32, nearly.

9. Required the approximate value of $\frac{64}{\sqrt{320}}$ to within ·000001.

Ans. 3·577781, nearly.

10. Required the approximate value of $\frac{8}{\sqrt[3]{32}}$ to within ·001.

Ans. 2·519.

11. Required the approximate value of $\frac{15}{\sqrt{15}}$ to within ·001.

Ans. 3·873, nearly.

12. Required the approximate value of $\frac{30}{\sqrt{60}}$ to within ·001.

Ans. 3·873, nearly.

13. Find the approximate value of $\frac{8}{\sqrt[3]{4}}$ to within ·01.

Ans. 5·04, nearly.

14. Find the approximate value of $\frac{2}{\sqrt{5}}$ to within ·01.

Ans. ·89.

15. Find the approximate value of $\frac{18}{\sqrt[3]{18}}$. *Ans.* 6·87.

CASE II.

317. To find a multiplier that will make rational an expression, consisting of a monomial surd, connected with rational terms, or consisting of two monomial surds.

Let it be required to make rational $\sqrt{p} + \sqrt{q}$ by multiplication. From the principle demonstrated in Case I., it is plain that $\sqrt{p}$ can only be made rational by multiplying it by $\sqrt{p}$, and $\sqrt{q}$ can only be made rational by multiplying it by $\sqrt{q}$. But, unless $\sqrt{p}$ and $\sqrt{q}$, in the multiplier, are connected by the sign minus, there will be two terms in the product remaining irrational and unreduced. Hence, the multiplier must have the minus sign between its terms.

Thus,
$$\begin{array}{l} \sqrt{p} + \sqrt{q} \\ \sqrt{p} - \sqrt{q} \\ \hline p + \sqrt{pq} \\ \quad - \sqrt{pq} - q \\ \hline p - q \end{array}$$

If the given expression is $\sqrt{p} - \sqrt{q}$, the multiplier must be, for a like reason, $\sqrt{p} + \sqrt{q}$.

If the given expression contain but one monomial surd, and is of the form $p + \sqrt{q}$, it may be written $\sqrt{p^2} + \sqrt{q}$, and reduced, as before, by multiplying by $\sqrt{p^2} - \sqrt{q}$. So, $p - \sqrt{q}$ may be written $\sqrt{p^2} - \sqrt{q}$, and may be made rational by multiplying it by $\sqrt{p^2} + \sqrt{q}$.

It matters not how many terms may be under the radical, or how many rational terms may be outside of the radical, the foregoing processes will still be applicable; because the sum of the quantities under the sign may be represented by a single letter, and the sum of the rational terms outside of the radical may be represented by a single letter. Thus, $a + b + \sqrt{m - n} = p + \sqrt{q} = \sqrt{p^2} + \sqrt{q}$. Multiply now by $\sqrt{p^2} - \sqrt{q}$, and replace p and q by their values. The result will be $(a + b)^2 - (m - n)$.

Let it be required to make $\sqrt[3]{p} + \sqrt[3]{q}$ rational by multiplication. The operation is as follows:

$$\begin{array}{l} \sqrt[3]{p} + \sqrt[3]{q} \\ \sqrt[3]{p^2} + \sqrt[3]{q^2} - \sqrt[3]{pq} \\ \hline p + \sqrt[3]{p^2q} \\ \quad + \sqrt[3]{pq^2} + q \\ \qquad - \sqrt[3]{p^2q} - \sqrt[3]{pq^2} \\ \hline p + q = \text{Product.} \end{array}$$

The multiplier, to make $\sqrt[3]{p}$ rational, must be $\sqrt[3]{p^2}$; and the multiplier, to make $\sqrt[3]{q}$ rational, must be $\sqrt[3]{q^2}$. But, after multiplication by $\sqrt[3]{p^2} + \sqrt[3]{q^2}$, there remained two uncancelled surds, $\sqrt[3]{p^2q}$ and $\sqrt[3]{q^2p}$, and these could only be cancelled by multiplying the given expression by $-\sqrt[3]{pq}$.

To make rational $\sqrt[3]{p} - \sqrt[3]{q}$, the multiplier must be $\sqrt[3]{p^2} + \sqrt[3]{q^2} + \sqrt[3]{pq}$. The multiplier, then, in every case, is the sum of the cube roots of the squares of the quantities diminished or augmented by the cube root of the product of the two quantities, according as the sign between the surds is plus or minus.

Let it be required to make $\sqrt[4]{p} + \sqrt[4]{q}$ rational. The operation is as follows:

$$\begin{array}{ll} \sqrt[4]{p} + \sqrt[4]{q} & = \text{Given surd,} \\ \sqrt[4]{p^3} - \sqrt[4]{q^3} - \sqrt[4]{p^2q} + \sqrt[4]{pq^2} & = \text{Multiplier,} \\ \hline p + \sqrt[4]{p^3q} & \\ \quad - \sqrt[4]{pq^3} - q & \\ \qquad - \sqrt[4]{p^3q} - \sqrt[4]{p^2q^2} & \\ \qquad\qquad + \sqrt[4]{p^2q^2} + \sqrt[4]{pq^3} & \\ \hline p \quad - \quad q \quad = \quad \text{Product.} & \end{array}$$

Or the operation may be performed in this manner:

$$\begin{array}{l} \sqrt[4]{p} + \sqrt[4]{q} = \text{Given surd,} \\ \sqrt[4]{p} - \sqrt[4]{q} = \text{First multiplier,} \\ \hline \sqrt[4]{p^2} - \sqrt[4]{q^2} = \sqrt{p} - \sqrt{q} = \text{First Product,} \\ \qquad\qquad \sqrt{p} + \sqrt{q} = \text{Second multiplier,} \\ \hline \qquad\qquad p \quad - \quad q = \text{Second product.} \end{array}$$

Any two monomial surds, whose common index is some power of 2, may be reduced in the same manner. And, since each multiplication gives a product containing two surds, with a common index one-half as great as the common index previous to multiplication, it is evident that the number of multiplications will be indicated by the exponent of the power of 2 in the primitive index, common to the surds given to be reduced.

Thus, $\sqrt[8]{p} + \sqrt[8]{q}$ can be reduced by three multiplications, since $8 = 2^3$. And $\sqrt[16]{p} + \sqrt[16]{q}$ can be reduced by four multiplications, since $16 = 2^4$.

The reduction of the $\sqrt[8]{p} + \sqrt[8]{q}$, and $\sqrt[16]{p} + \sqrt[16]{q}$, is as follows:

$$
\begin{aligned}
&\sqrt[8]{p} + \sqrt[8]{q} = \text{Given surd,}\\
&\sqrt[8]{p} - \sqrt[8]{q} = \text{First multiplier,}\\
\hline
&\sqrt[8]{p^2} - \sqrt[8]{q^2} = \sqrt[4]{p} - \sqrt[4]{q} = \text{First product,}\\
&\qquad\qquad\qquad\quad \sqrt[4]{p} + \sqrt[4]{q} = \text{Second multiplier,}\\
&\qquad\qquad\qquad\quad \sqrt[4]{p^2} - \sqrt[4]{q^2} = \sqrt{p} - \sqrt{q} = \text{Second product,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \sqrt{p} + \sqrt{q} = \text{Third multiplier,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad p - q = \text{Third product.}
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt[16]{p} + \sqrt[16]{q} = \text{Given surd,}\\
&\sqrt[16]{p} - \sqrt[16]{q} = \text{First multiplier,}\\
\hline
&\sqrt[16]{p^2} - \sqrt[16]{q} = \sqrt[8]{p} - \sqrt[8]{q} = \text{First product,}\\
&\qquad\qquad\qquad\quad \sqrt[8]{p} + \sqrt[8]{q} = \text{Second multiplier,}\\
&\qquad\qquad\qquad\quad \sqrt[8]{p^2} - \sqrt[8]{q^2} = \sqrt[4]{p} - \sqrt[4]{q} = \text{Second product,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \sqrt[4]{p} + \sqrt[4]{q} = \text{Third multiplier,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \sqrt{p} - \sqrt{q} = \text{Third product,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \sqrt{p} + \sqrt{q} = \text{Fourth multiplier,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad p - q = \text{Fourth product.}
\end{aligned}
$$

We have this process for the reduction of $\sqrt[5]{p} + \sqrt[5]{q}$.

$$
\begin{aligned}
&\sqrt[5]{p} + \sqrt[5]{q} = \text{Given surd,}\\
&\sqrt[5]{p^4} + \sqrt[5]{q^4} - \sqrt[5]{p^3q} - \sqrt[5]{pq^3} + \sqrt[5]{p^2q^2} = \text{Multiplier,}\\
\hline
&p + \sqrt[5]{p^4q}\\
&\quad + \sqrt[5]{pq^4} + q\\
&\qquad\qquad - \sqrt[5]{p^4q} - \sqrt[5]{p^3q^2}\\
&\qquad\qquad\qquad - \sqrt[5]{p^2q^3} - \sqrt[5]{pq^4}\\
&\qquad\qquad\qquad\qquad + \sqrt[5]{p^3q^2} + \sqrt[5]{p^2q^3}\\
\hline
&p + q \qquad\qquad\qquad\qquad\qquad = \text{Product.}
\end{aligned}
$$

The $\sqrt[6]{p}+\sqrt[6]{q}$ can be reduced in a similar manner.

$$\begin{array}{rl}
\sqrt[6]{p}+\sqrt[6]{q} = & \text{Given expression,} \\
\sqrt[6]{p}-\sqrt[6]{q} = & \text{First multiplier,} \\
\hline
\sqrt[6]{p^2}+\sqrt[6]{q^2} = \sqrt[3]{p}-\sqrt[3]{q} = & \text{First product,} \\
\sqrt[3]{p^2}-\sqrt[3]{q^2}+\sqrt[3]{pq} = & \text{Second multiplier,} \\
\hline
p - q = & \text{Second product.}
\end{array}$$

All expressions composed of two surds, whose common index is a multiple of 2 and 3, may be reduced in a similar manner.

Thus,

$$\begin{array}{rl}
\sqrt[12]{p}+\sqrt[12]{q} = & \text{Given expression,} \\
\sqrt[12]{p}-\sqrt[12]{q} = & \text{First multiplier,} \\
\hline
\sqrt[6]{p}-\sqrt[6]{q} = & \text{First product,} \\
\sqrt[6]{p}+\sqrt[6]{q} = & \text{Second multiplier,} \\
\hline
\sqrt[3]{p}-\sqrt[3]{q} = & \text{Second product,} \\
\sqrt[3]{p}-\sqrt[3]{q}+\sqrt[3]{pq} = & \text{Third multiplier,} \\
\hline
p - q = & \text{Third product.}
\end{array}$$

All expressions composed of two surds, whose common index is some power of 3, may be reduced in the same manner as $\sqrt[3]{p}+\sqrt[3]{q}$.

Take as an example

$$\begin{array}{rl}
\sqrt[9]{p}+\sqrt[9]{q} = & \text{Given expression,} \\
\sqrt[9]{p^2}+\sqrt[9]{q^2}-\sqrt[9]{pq} = & \text{First multiplier,} \\
\hline
\sqrt[9]{p^3}+\sqrt[9]{q^3} = \sqrt[3]{p}+\sqrt[3]{q} = & \text{First product,} \\
\sqrt[3]{p^2}+\sqrt[3]{q^2}-\sqrt[3]{pq} = & \text{Second multiplier,} \\
\hline
p + q = & \text{Second product.}
\end{array}$$

Take as a second example, $\sqrt[27]{p}+\sqrt[27]{q}$.

Then,

$$\begin{array}{rl}
\sqrt[27]{p}+\sqrt[27]{q} = & \text{Given expression,} \\
\sqrt[27]{p^2}+\sqrt[27]{q^2}-\sqrt[27]{pq} = & \text{First multiplier,} \\
\hline
\sqrt[27]{p^3}+\sqrt[27]{q^3} = \sqrt[9]{p}+\sqrt[9]{q} = & \text{First product, which can be reduced as before.}
\end{array}$$

The reduction of $\sqrt[7]{p}+\sqrt[7]{q}$, is more difficult than any of the preceding reductions.

We will write the terms of the product that cancel each other in the same vertical column.

$$\sqrt[7]{p}+\sqrt[7]{q} = \text{Given expression,}$$
$$\sqrt[7]{p^6}+\sqrt[7]{q^6}-\sqrt[7]{p^5q}-\sqrt[7]{pq^5}+\sqrt[7]{p^4q^2}+\sqrt[7]{q^4p^2}-\sqrt[7]{p^3q^3} = \text{Multiplier.}$$
$$\begin{array}{l} p+\sqrt[7]{p^6q}+q+\sqrt[7]{pq^6}-\sqrt[7]{p^5q^2}-\sqrt[7]{p^2q^5}+\sqrt[7]{p^4q^3}+\sqrt[7]{p^3q^4} \\ \quad -\sqrt[7]{p^6q} \quad -\sqrt[7]{pq^6}+\sqrt[7]{p^5q^2}+\sqrt[7]{p^2q^5}-\sqrt[7]{p^4q^3}-\sqrt[7]{p^3q^4}, \\ \hline p+q \qquad\qquad = \text{Product.} \end{array}$$

All expressions composed of two monomial surds, may be rendered rational when the surds have a common index. The amount of difficulty attending the reduction depends altogether upon the index. When the index is some power of 2, the reduction is very easy. But when it is 7, 11, 13, 17, &c., the reduction is difficult.

When an expression is given to be reduced, we must first examine the factors of the common index, and make our reduction correspond to those factors.

Thus, let it be required to render rational $\sqrt[15]{p}+\sqrt[15]{q}$. The factors of 15 are 3 and 5; we must then reduce the expression by the first multiplication, so that the common index of the result shall be 5.

Thus, $\sqrt[15]{p}+\sqrt[15]{q} = \text{Given expression,}$

$\sqrt[15]{p^2}+\sqrt[15]{q^2}-\sqrt[15]{pq} = \text{First multiplier,}$

$\sqrt[15]{p^3}+\sqrt[15]{q^3} = \sqrt[5]{p}+\sqrt[5]{q}$, which can be reduced as before.

To reduce the $\sqrt[10]{p}+\sqrt[10]{q}$, we must use such a multiplier as will leave a common index, 5, in the product.

Thus, $\sqrt[10]{p}+\sqrt[10]{q}$,

$\sqrt[10]{p}-\sqrt[10]{q}$,

$\sqrt[10]{p^2}-\sqrt[10]{q^2} = \sqrt[5]{p}-\sqrt[5]{q}$, which can be made rational as before.

Let it be required to reduce $\sqrt[24]{p}+\sqrt[24]{q}$.

The factors of the index are $3\,(2)^3$. Therefore, we must first get rid of the factor, 3, and reduce the expression to a common index, $(2)^3$ or 8.

$$\begin{aligned}
&\sqrt[24]{p} + \sqrt[24]{q} = \text{Given expression,}\\
&\sqrt[24]{p^2} + \sqrt[24]{q^2} - \sqrt[24]{pq} = \text{First multiplier,}\\
&\sqrt[8]{p} + \sqrt[8]{q} = \text{First product,}\\
&\sqrt[8]{p} - \sqrt[8]{q} = \text{Second multiplier,}\\
&\sqrt[4]{p} - \sqrt[4]{q} = \text{Second product,}\\
&\sqrt[4]{p} + \sqrt[4]{q} = \text{Third multiplier,}\\
&\sqrt{p} - \sqrt{q} = \text{Third product,}\\
&\sqrt{p} + \sqrt{q} = \text{Fourth multiplier,}\\
&p - q = \text{Fourth product.}
\end{aligned}$$

Corollary.

318. The principles developed in Case II. enable us to find the approximate value of a fraction, whose denominator consists of a monomial surd connected with known terms, or of two monomial surds.

Let it be required to find the approximate value of $\frac{2}{\sqrt{5} + \sqrt{2}}$.

Then, $\frac{2}{\sqrt{5} + \sqrt{2}} = \frac{2(\sqrt{5} - \sqrt{2})}{5 - 2} = \frac{2(2{\cdot}23 - 1.41)}{3} = \frac{2(0{\cdot}82)}{3} = \frac{1{\cdot}64}{3}$ to within ·01.

EXAMPLES.

1. Required the approximate value of $\frac{3}{\sqrt{8} + \sqrt{5}}$ to within ·01.

Ans. ·59.

2. Required the approximate value of $\frac{3}{8 + \sqrt{5}}$ to within ·01.

Ans. $\frac{17{\cdot}31}{59}$.

For $\frac{3}{8 + \sqrt{5}} = \frac{3(8 - \sqrt{5})}{64 - 5} = \frac{3(8 - 2{\cdot}23)}{59} = \frac{3(5{\cdot}77)}{59} = \frac{17{\cdot}31}{59}$.

3. Required the approximate value of $\frac{66}{4 + \sqrt[3]{2}}$ to within ·1.

Ans. 12·6

For $\frac{66}{4+\sqrt[3]{2}} = \frac{66}{\sqrt[3]{4^3}+\sqrt[3]{2}} = \frac{66.(\sqrt[3]{(4)^6}+\sqrt[3]{(2)^2}-\sqrt[3]{(4)^3.2})}{(\sqrt[3]{4^3}+\sqrt[3]{2})(\sqrt[3]{(4)^6}+\sqrt[3]{(2)^2}-\sqrt[3]{(4^3)}\cdot 2)}$
$= \frac{66\ ((4)^2+1\cdot6-5)}{(4)^3+2} = 12\cdot6.$

4. Required the approximate value of $\frac{2}{\sqrt[3]{4}+\sqrt[3]{2}}$ to within ·1.

Ans. ·7.

For $\frac{2}{\sqrt[3]{4}+\sqrt[3]{2}} = \frac{2(\sqrt[3]{(4)^2}+\sqrt[3]{(2)^2}-\sqrt[3]{4\cdot2})}{4+2} = \frac{2(2\cdot5+1\cdot6-2}{6}$
$= \frac{4\cdot2}{6} = \cdot7.$

5. Required the approximate value of $\frac{116}{11-\sqrt{5}}$ to within ·01.

Ans. 13·23.

6. Required the approximate value of $\frac{6}{\sqrt{11}+\sqrt{5}}$ to within ·01.

Ans. 1·08.

7. Required the approximate value of $\frac{60}{\sqrt[3]{16}+\sqrt[3]{4}}$ to within ·1.

Ans. 14·4.

8. Required the approximate value of $\frac{30}{\sqrt[3]{25}+\sqrt[3]{5}}$ to within ·1.

Ans. 6·4.

9. Required the approximate value of $\frac{8}{\sqrt{6}-\sqrt{5}}$ to within ·001.

Ans. 37·44.

10. Required the approximate value of $\frac{2}{\sqrt[3]{4}-\sqrt[3]{2}}$ to within ·1.

Ans. 6·1.

CASE III.

To make rational an expression containing three or more terms of the square root.

319. Let it be required to make rational $\sqrt{p}+\sqrt{q}+\sqrt{n}$.

The process is as follows:

$$\begin{array}{l}
\sqrt{p} + \sqrt{q} + \sqrt{n} = \text{Given expression,} \\
\sqrt{p} - \sqrt{q} + \sqrt{n} = \text{First multiplier,} \\
\hline
p + \sqrt{pq} + \sqrt{pn} \\
\quad - \sqrt{pq} - \sqrt{qn} - q \\
\qquad + \sqrt{pn} + \sqrt{qn} + n \\
\hline
p - q + n + 2\sqrt{pn} = \text{First product,} \\
p - q + n - 2\sqrt{pn} = \text{Second multiplier,} \\
\hline
(p - q + n)^2 - 4pn = \text{Second product.}
\end{array}$$

Let it be required to make rational $\sqrt{p} + \sqrt{q} - \sqrt{n}$.
The process is as follows:

$$\begin{array}{l}
\sqrt{p} + \sqrt{q} - \sqrt{n} = \text{Given expression,} \\
\sqrt{p} + \sqrt{q} + \sqrt{n} = \text{First multiplier,} \\
\hline
p + \sqrt{pq} - \sqrt{pn} \\
\quad + \sqrt{pq} + q - \sqrt{qn} \\
\qquad + \sqrt{pn} + \sqrt{qn} - n \\
\hline
p + q - n + 2\sqrt{pq} = \text{First product,} \\
p + q - n - 2\sqrt{pq} = \text{Second multiplier,} \\
\hline
(p + q - n)^2 - 4pq = \text{Second product.}
\end{array}$$

Take as a third example $\sqrt{p} - \sqrt{q} + \sqrt{m} + \sqrt{n}$.

$$\begin{array}{l}
\sqrt{p} - \sqrt{q} + \sqrt{m} + \sqrt{n} \\
\sqrt{p} + \sqrt{q} - \sqrt{m} + \sqrt{n} \\
\hline
p - \sqrt{pq} + \sqrt{pm} + \sqrt{pn} \\
n \qquad + \sqrt{pn} - \sqrt{nq} + \sqrt{nm} \\
-q + \sqrt{pq} \qquad + \sqrt{nq} \qquad + \sqrt{qm} \\
-m \qquad - \sqrt{pm} \qquad - \sqrt{mn} + \sqrt{qm} \\
\hline
(p + n - q - m) + 2\sqrt{pn} + 2\sqrt{qm} = \text{First product,} \\
(p + n - q - m) - 2\sqrt{pn} + 2\sqrt{qm} = \text{Second multiplier,} \\
\hline
P^2 + 4P\sqrt{qm} - 4pn + 4qm = \text{Second product reduced, and in which P represents the rational term.}
\end{array}$$

We may represent $P^2 - 4pn + 4qm$ by M^2, and then we will have

$$\begin{aligned} M^2 + 4P\sqrt{qm} &= \text{Second product,} \\ M^2 - 4P\sqrt{qm} &= \text{Third multiplier,} \\ \hline M^4 - 16P^2qm &= \text{Third product.} \end{aligned}$$

Corollary.

320. The principles of Case III. enable us to approximate to the value of a fraction containing three or more monomial surds.

EXAMPLES.

1. Reduce $\dfrac{10}{\sqrt{5} + \sqrt{6} + 1}$ to an equivalent fraction having a rational denominator. *Ans.* $5 + \sqrt{5} - \sqrt{30}$.

For $\dfrac{10}{\sqrt{5} + \sqrt{6} + 1} = \dfrac{10(\sqrt{5} - \sqrt{6} + 1)}{(\sqrt{5} + \sqrt{6} + 1)(\sqrt{5} - \sqrt{6} + 1)} =$
$\dfrac{10(\sqrt{5} - \sqrt{6} + 1)}{2\sqrt{5}} = \dfrac{10\sqrt{5}(\sqrt{5} - \sqrt{6} + 1)}{2(5)} = 5 - \sqrt{30} + \sqrt{5}.$

The approximate value of the result can be found, if desired.

2. Reduce $\dfrac{2}{\sqrt{6} + \sqrt{8} + \sqrt{14}}$ to an equivalent fraction with a rational denominator. *Ans.* $\dfrac{(\sqrt{6} - \sqrt{8} + \sqrt{14}(6 - \sqrt{84})}{-48}$.

3. Reduce $\dfrac{20}{1 + \sqrt{5} + \sqrt{6} + \sqrt{10}}$ to an equivalent fraction with a rational denominator.

Ans. $\dfrac{(1 - \sqrt{5} - \sqrt{6} + \sqrt{10})(\sqrt{10} + \sqrt{30})}{-2}$.

4. Reduce $\dfrac{-8}{\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}}$ to an equivalent fraction with a rational denominator. *Ans.* $(3 - \sqrt{2} - \sqrt{3})(4 + 2\sqrt{6})$.

5. Reduce $\dfrac{2}{\sqrt{3}+\sqrt{5}+\sqrt{6}+\sqrt{8}}$ to an equivalent fraction with a rational denominator.

$$Ans.\ \frac{(\sqrt{3}-\sqrt{5}-\sqrt{6}+\sqrt{8})(\sqrt{24}+\sqrt{30})}{-6}.$$

6. Reduce $\dfrac{2}{\sqrt{3}+\sqrt{5}+\sqrt{6}+\sqrt{10}}$ to an equivalent fraction.

Ans. $\sqrt{3}-\sqrt{5}-\sqrt{6}+\sqrt{10}$.

EXTRACTION OF THE SQUARE ROOT OF A MONOMIAL SURD CONNECTED WITH A RATIONAL TERM, OR OF TWO MONOMIAL SURDS.

321. Before we proceed to extract the root, it will be necessary to demonstrate three principles upon which the extraction depends.

First Principle.

The square root of a quantity cannot consist of the sum of two parts, one of which is rational and the other irrational.

For, if possible, suppose $\sqrt{a} = x + \sqrt{y}$. Then, by squaring both members, we get $a = x^2 + 2x\sqrt{y} + y$. From which, $\sqrt{y} = \dfrac{a - x^2 - y}{2x}$. That is an irrational quantity equal to a rational one, which is absurd. But the absurdity has not resulted from an error in the analysis, and must, therefore, be in the condition.

Second Principle.

When the first member of any equation contains a monomial surd, connected with rational terms, and the second member is made up in the same manner, the rational quantities in the first member are equal to the rational quantities in the second, and the irrational quantities in the two members are respectively equal also.

Let $a + \sqrt{b} = x + \sqrt{y}$, then $a = x$, and $\sqrt{b} = \sqrt{y}$.

For if a be not equal to x, let $a = x \pm m$. Substitute this for a in the equation, and there results $x \pm m + \sqrt{b} = x + \sqrt{y}$, or $\pm m + \sqrt{b} = \sqrt{y}$, which is impossible, (principle first.) Therefore, it is absurd to suppose that a is unequal to x. Hence, $a = x$, and the equation becomes $a + \sqrt{b} = a + \sqrt{y}$, or by cancelling a, $\sqrt{b} = \sqrt{y}$.

Third Principle.

If $\sqrt{a+\sqrt{b}}=x+\sqrt{y}$, then will $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$.

For, by squaring the first equation, we get $a+\sqrt{b}=x^2+2x\sqrt{y}+y$ (1).

From which $a=x^2+y$ (2), and $\sqrt{b}=2x\sqrt{y}$ (3).

Subtract (3) from (2), and we have $a-\sqrt{b}=x^2-2x\sqrt{y}+y$ (4).

Extracting the square root of both members of (4), we have $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$.

This last equation has been correctly deduced from the equation $\sqrt{a+\sqrt{b}}=x+\sqrt{y}$, so that it is a true equation, provided that the equation from which it is deduced is true.

322. The foregoing principles enable us to deduce a formula for the extraction of the square root of an expression made up of two monomial surds, or of one monomial surd connected with rational terms.

Let, $$\sqrt{a+\sqrt{b}}=x+\sqrt{y}. \quad (1).$$

Then, $$\sqrt{a-\sqrt{b}}=x-\sqrt{y}. \quad (2).$$

Squaring (1) and (2), there results

$$a+\sqrt{b}=x^2+2x\sqrt{y}+y. \quad (3).$$
$$a-\sqrt{b}=x^2-2x\sqrt{y}+y. \quad (4).$$

Adding (3) and (4), and cancelling (2) in the result, we get $a=x^2+y$. (5).

Multiplying (1) and (2), we get $\sqrt{a^2-b}=x^2-y$. (6).

Adding (5) and (6), we get $\dfrac{a+\sqrt{a^2-b}}{2}=x^2$. (7).

Subtracting (6) from (5), we get $\dfrac{a-\sqrt{a^2-b}}{2}=y$. (8).

Extracting the square roots of both members of (7) and (8), and there results $$\sqrt{\frac{a+\sqrt{a^2-b}}{2}}=x. \quad (9).$$

and $$\sqrt{\frac{a-\sqrt{a^2-b}}{2}} = \sqrt{y}. \quad (10).$$

By adding (9) and (10), we get the value of $x + \sqrt{y}$, or of the equivalent, $\sqrt{a+\sqrt{b}}$.

By subtracting (10) from (9), we get the value of $x - \sqrt{y}$, or the equivalent $\sqrt{a-\sqrt{b}}$.

Hence we have, by performing these operations, the two formulæ,

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}. \quad (A).$$

$$\sqrt{a-\sqrt{b}} = \sqrt{\frac{a+\sqrt{a_2-b}}{2}} - \sqrt{\frac{a-\sqrt{a^2-b}}{2}}. \quad (B).$$

By using the double sign $\pm$, we may unite (A) and (B) in one formula.

$$\sqrt{a\pm\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a+\sqrt{a^2-b}}{2}}. \quad (C).$$

We will now show the use of the formulæ.

EXAMPLES.

1. Let it be required to extract the square root of $9 + \sqrt{45}$. Then, by comparing $9 + \sqrt{45}$ with $a + \sqrt{b}$, the quantity whose root is to be extracted in (A), we see that $a = 9$ and $b = 45$.

Hence, $\sqrt{a+\sqrt{b}} = \sqrt{9+\sqrt{45}} = \sqrt{\frac{9+\sqrt{81-45}}{2}} + \sqrt{\frac{9-\sqrt{81-45}}{2}} = \sqrt{7\frac{1}{2}} + \sqrt{\frac{3}{2}}.$

A result that can be verified; for, squaring it, we will have, $7\frac{1}{2} + 2\sqrt{\frac{45}{4}} + \frac{3}{2} = 9 + \sqrt{45}$.

2. Required the square root of $9 + \sqrt{72}$. By comparison, we get $a = 9$ and $b = 72$.

And from (A) we get $\sqrt{9 + \sqrt{72}} = \sqrt{\frac{9 + \sqrt{81 - 72}}{2}} + \sqrt{\frac{9 - \sqrt{81 - 72}}{2}} = \sqrt{6} + \sqrt{3}.$

This result can be verified; for, squaring it, we get $6 + 2\sqrt{18} + 3 = 9 + \sqrt{72}.$

3. Required the square root of $9 - \sqrt{56}.$

Then, $a = 9$ and $b = 56$. Formula (B) must be used.

From (B) we get $\sqrt{9 - \sqrt{56}} = \sqrt{\frac{9 + \sqrt{81 - 56}}{2}} - \sqrt{\frac{9 - \sqrt{81 - 56}}{2}} = \sqrt{7} - \sqrt{2}.$

Squaring this root, we have, $7 - 2\sqrt{14} + 2 = 9 - \sqrt{56}.$

4. Required the square root of $9 + 2\sqrt{-162}.$

Pass the 2 under the radical, thus, $9 + 2\sqrt{-162} = 9 + \sqrt{-648},$ and $\sqrt{a^2 - b} = \sqrt{81 + 648} = \sqrt{729} = 27$, and the application of the formula will give $\sqrt{9 + 2\sqrt{-162}} = \sqrt{18} + \sqrt{-9}.$

This result, when squared, will give $18 + 2\sqrt{-162} - 9 = 9 + 2\sqrt{-162}.$ Hence, the result is true.

5. Required the sum of $\sqrt{9 + 2\sqrt{-162}}$, and $\sqrt{9 - 2\sqrt{-162}}.$

Ans. $2\sqrt{18}.$

This result can be verified; for, $2\sqrt{18}$ squared $= 72$; and the given expression, when squared, produces $9 + 2\sqrt{-162} + 9 - 2\sqrt{-162} + 2\sqrt{81 + 648} = 18 + 2\sqrt{729} = 18 + 54 = 72.$

6. Required the square root of $\sqrt{8} + \sqrt{-1}.$

Ans. $\sqrt{\frac{\sqrt{8} + 3}{2}} + \sqrt{\frac{\sqrt{8} - 3}{2}}.$

This result can be verified by squaring it. We will get $\frac{\sqrt{8} + 3}{2} + \frac{\sqrt{8} - 3}{2} + 2\sqrt{\frac{8 - 9}{4}} = \sqrt{8} + \sqrt{-1}.$

7. Required the square root of $1 + \sqrt{-15}$.

Ans. $\sqrt{\frac{5}{2}} + \sqrt{-\frac{3}{2}}$.

Verify the result by squaring it.

8. Required the square root of $1 + 2\sqrt{-20}$.

Ans. $\sqrt{5} + \sqrt{-4}$.

Verify the result by squaring it.

9. Required the square root of $1 + \sqrt{-288}$.

Ans. $3 + \sqrt{-8}$.

Verify the result by squaring it.

10. Required the sum of $\sqrt{1+\sqrt{-288}}$, and $\sqrt{1-\sqrt{-288}}$.

Ans. 6.

We see that the square root of an expression containing an imaginary quantity, cannot have all its terms real. But the sum of the roots of two expressions, involving imaginary quantities, may be real and even rational. The formulas (A) and (B) are generally applied to expressions for which $\sqrt{a^2-b}$ is rational.

11. Reduce $\frac{10}{\sqrt[4]{9} + \sqrt[4]{4}}$ to an equivalent fraction with a rational denominator. *Ans.* $10(\sqrt{3} - \sqrt{2})$.

For, $\frac{10}{\sqrt[4]{9} + \sqrt[4]{4}} = 10\,\frac{(\sqrt[4]{(9)^3} - \sqrt[4]{(4)^3} - \sqrt[4]{(9)^2 4} + \sqrt[4]{9(4)^2})}{9-4} =$

$= 2(\sqrt{27} - \sqrt{8} - \sqrt{18} + \sqrt{12}) = 2(3\sqrt{3} - 2\sqrt{2} - 3\sqrt{2} + 2\sqrt{3})$

$= 2(5\sqrt{3} - 5\sqrt{2}) = 10(\sqrt{3} - \sqrt{2})$.

Or, the reduction may be performed thus, $\frac{10}{\sqrt[4]{9} + \sqrt[4]{4}} = \frac{10}{\sqrt{3} + \sqrt{2}}$

$= \frac{10(\sqrt{3} - \sqrt{2})}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} = \frac{10(\sqrt{3} - \sqrt{2})}{3-2} = 10(\sqrt{3} - \sqrt{2})$.

12. Reduce $\frac{4}{\sqrt[4]{5} + \sqrt[4]{4}}$ to an equivalent fraction with a rational denominator. *Ans.* $4((2 + \sqrt{5})(\sqrt[4]{5}) - 2\sqrt{2} - \sqrt{10})$.

For, $\frac{4}{\sqrt[4]{5}+\sqrt[4]{4}} = 4\,\frac{(\sqrt[4]{5^3}-\sqrt[4]{(4)^3}-\sqrt[4]{(5)^2 4}+\sqrt[4]{5(4)^2})}{5-4} =$
$4\,(\sqrt{5}\,\sqrt[4]{5}-\sqrt{8}-\sqrt{10}+2\sqrt[4]{5}) = 4\,(\,(2+\sqrt{5})(\sqrt[4]{5})-2\sqrt{2}-\sqrt{10}).$

13. Reduce $\frac{5}{\sqrt[3]{4}+\sqrt[3]{2}}$ to an equivalent fraction with a rational denominator. *Ans.* $\frac{5}{6}\,(\sqrt[3]{4}+(\sqrt[3]{2}-1)2).$

For, $\frac{5}{\sqrt[3]{4}+\sqrt[3]{2}} = 5\,\frac{(\sqrt[3]{(4)^2}+\sqrt[3]{(2)^2}-\sqrt[3]{(4)\,(2)}}{(\sqrt[3]{4}+\sqrt[3]{2})\,(\sqrt[3]{(4)^2}+\sqrt[3]{(2)^2}-\sqrt[3]{(4)\,(2)})}$
$= \frac{5}{6}\,(2\sqrt[3]{2}+\sqrt[3]{4}-2) = \frac{5}{6}\,(\sqrt[3]{4}+(\sqrt[3]{2}-1)2).$

14. Reduce $\frac{6}{\sqrt[3]{27}+\sqrt[3]{8}}$ to an equivalent fraction with a rational denominator. *Ans.* $\frac{6}{5}$.

For, $\frac{6}{\sqrt[3]{27}+\sqrt[3]{8}} = 6\,\frac{(\sqrt[3]{(27)^2}+\sqrt[3]{(8)^2}-\sqrt[3]{(27)\,(8)}\,)}{35} =$
$\frac{6}{35}\,(9+4-6) = \frac{42}{35} = \frac{6}{5},$

The result is rational because the given expression was really rational, though under an irrational form.

15. Reduce $\frac{m}{n+\sqrt{m}}$ to an equivalent fraction with a rational denominator. *Ans.* $\frac{m\,(n-\sqrt{m})}{n^2-m}$.

IMAGINARY QUANTITIES.

323. Any expression whatever, made up of monomial surds and rational terms, or monomial surds only, may be rendered rational by repeated multiplications. The few examples given will show the manner in which these multiplications must be made. No general rule can be given in regard to them. There is a particular class of monomial surds, which, when rendered rational, present some differences from other surds in regard to their algebraic signs. These are imaginary surds.

An imaginary quantity has been defined to be the even root of a negative quantity, because no quantity, taken as a factor an even number of times, can give a negative result. Thus, $\sqrt{-a}$, $\sqrt[4]{-a}$, $\sqrt[6]{-a}$, $\sqrt[2m]{-a}$, are imaginary quantities.

If the indicated root be of the $2m^{th}$ degree, then $\sqrt[2m]{-a}$ may be written $\sqrt[2m]{a}\sqrt[2m]{-1} = b\sqrt{-1}$. In which, b represents the $2m^{th}$ root of a, whether that be rational or irrational. And we see that all imaginary quantities may be decomposed into two factors, one of which is an indicated even root of minus unity, and the other of which is real, and sometimes rational.

This decomposition must always be first effected previous to operating upon imaginary quantities.

To square $\sqrt{-a}$, we first write the expression $\sqrt{a}\ \sqrt{-1}$; these two factors will both drop the radical when squared. Hence, $(\sqrt{-a})^2 = (\sqrt{a}\ \sqrt{-1})^2 = (a)\ (-1) = -a$.

So, also, $(\sqrt{-a})^3 = (\sqrt{a}\sqrt{-1})^3 = (a\sqrt{a})\ (-1\ \sqrt{-1}) = -a\sqrt{-a}$.

$$(\sqrt{-a})^4 = (\sqrt{a}\ \sqrt{-1})^4 = (a^2)\ (+1) = +\ a^2.$$

$$(\sqrt{-a})^5 = (\sqrt{a}\ \sqrt{-1})^5 = (a^2\ \sqrt{a})\ (+\sqrt{-1}) = a^2\sqrt{-a}.$$

$$(\sqrt{-a})^6 = (\sqrt{a}\ \sqrt{-1})^6 = (a^3)\ (-1) = -\ a^3.$$

The table shows that the even powers of imaginary monomials are always real, and that their signs are alternately plus and minus.

324. A table of products will show the modifications of their algebraic signs.

$(+\sqrt{-a})\ (+\sqrt{-a}) = +(\sqrt{a}\ \sqrt{-1})\ (\sqrt{a}\ \sqrt{-1}) = (\sqrt{a})\ (\sqrt{a})\ (\sqrt{-1})\ (\sqrt{-1}) = -\ a.$

$(-\sqrt{-a})\ (-\sqrt{-a}) = (-\sqrt{a}\ \sqrt{-1})\ (-\sqrt{a}\ \sqrt{-1}) = (-\sqrt{a})\ (-\sqrt{a})\ (\sqrt{-1})\ (\sqrt{-1}) = -\ a.$

$(+\sqrt{-a})\ (-\sqrt{-a}) = (\sqrt{a}\ \sqrt{-1})\ (-\sqrt{a}\ \sqrt{-1}) = (\sqrt{a})\ (-\sqrt{a})\ (\sqrt{-1})\ (\sqrt{-1}) = +\ a.$

$(+\sqrt{-a})\ (+\sqrt{-b}) = (\sqrt{a}\ \sqrt{-1})\ (\sqrt{b}\ \sqrt{-1}) = (\sqrt{a})\ (\sqrt{b})\ (\sqrt{-1})\ (\sqrt{-1}) = -\sqrt{ab}.$

$(-\sqrt{-a})\ (-\sqrt{-b}) = (-\sqrt{a}\ \sqrt{-1})\ (-\sqrt{b}\ \sqrt{-1}) = (-\sqrt{a})\ (-\sqrt{b})\ (\sqrt{-1})\ (\sqrt{-1}) = -\sqrt{ab}.$

$(+\sqrt{-a})\ (-\sqrt{-b}) = (\sqrt{a}\ \sqrt{-1})\ (-\sqrt{b}\ \sqrt{-1}) = (\sqrt{a})$ $(-\sqrt{b})\ (\sqrt{-1})\ (\sqrt{-1}) = +\sqrt{ab}.$

We see that like signs produce minus, and unlike signs produce plus, when imaginary monomials are multiplied together.

325. We will now form a table of quotients.

$$\frac{+\sqrt{-a}}{+\sqrt{-b}} = \frac{\sqrt{a}\ \sqrt{-1}}{\sqrt{b}\ \sqrt{-1}} = \frac{+\sqrt{a}}{+\sqrt{b}}.$$

$$\frac{+\sqrt{-a}}{-\sqrt{-b}} = \frac{\sqrt{a}\ \sqrt{-1}}{-\sqrt{b}\ \sqrt{-1}} = -\frac{\sqrt{a}}{\sqrt{b}}.$$

$$\frac{-\sqrt{-a}}{-\sqrt{-b}} = \frac{-\sqrt{a}\ \sqrt{-1}}{-\sqrt{-b}\ \sqrt{-1}} = \frac{-\sqrt{a}}{-\sqrt{b}} = +\frac{\sqrt{a}}{\sqrt{b}}.$$

$$\frac{-\sqrt{-a}}{+\sqrt{-b}} = \frac{-\sqrt{a}\ \sqrt{-1}}{\sqrt{b}\ \sqrt{-1}} = \frac{-\sqrt{a}}{\sqrt{b}}.$$

And we see that, in division, like signs produce plus, and unlike signs produce minus. It would seem that, since the rule for the signs in division is different from that in multiplication, the product of the quotient by the divisor might not give the dividend. But any of the preceding quotients can be readily verified.

Take $\dfrac{\sqrt{-a}}{\sqrt{-b}} = \dfrac{+\sqrt{a}}{+\sqrt{b}}.$

Multiply the quotient $\dfrac{+\sqrt{a}}{+\sqrt{b}}$ by the divisor $\sqrt{-b}$. Then, $\dfrac{+\sqrt{a}}{+\sqrt{b}} \times \sqrt{-b} = \dfrac{+\sqrt{a}}{+\sqrt{b}}\sqrt{b}\ \sqrt{-1} = +\sqrt{a}\ \sqrt{-1} = \sqrt{-a}$, the dividend.

326. Imaginary quantities can be operated upon just as real quantities, provided that care be given to attribute the proper algebraic signs to the results.

Let it be required to divide $4\sqrt{-b^2}$ by $2b\sqrt{-1}$. The division cannot be performed until $4\sqrt{-b^2}$ is transformed into the equivalent expression, $4b\sqrt{-1}$, the quotient will then be 2.

Let it be required to multiply $4\sqrt{-b^2}$ by $2b\sqrt{-1}$. Then, $4\sqrt{-b^2} \times 2b\sqrt{-1} = 4b\sqrt{-1} \times 2b\sqrt{-1} = -8b^2.$

Now let it be required to divide $-8b^2$ by $2b\sqrt{-1}$. Then, $\frac{-8b^2}{2b\sqrt{-1}} = \frac{8b^2\sqrt{-1}\sqrt{-1}}{2b\sqrt{-1}} = -4b\sqrt{-1}$.

EXAMPLES.

1. Multiply $+\sqrt{-a}$ by $-\sqrt{-b}$. *Ans.* $+\sqrt{ab}$.

2. Divide $\sqrt{ab}$ by $\sqrt{-a}$. *Ans.* $-\sqrt{-b}$.

For, $\frac{\sqrt{ab}}{\sqrt{-a}} = \frac{\sqrt{a}\sqrt{b}}{\sqrt{a}\sqrt{-1}} = \frac{\sqrt{b}}{\sqrt{-1}} = \frac{-\sqrt{b}\sqrt{-1}\sqrt{-1}}{\sqrt{-1}} = -\sqrt{b}\sqrt{-1} = -\sqrt{-b}$.

The minus sign is placed before the square root of b, in the expression, $-\sqrt{b}\sqrt{-1}\sqrt{-1}$, in order to make the result positive. Since $\sqrt{-1}\sqrt{-1} = -1$.

3. Divide $\sqrt{ab}$ by $-\sqrt{-b}$. *Ans.* $+\sqrt{-a}$.

4. Multiply $ab\sqrt{-c^2}$ by $\sqrt{-a^2bc}$. *Ans.* $-a^2bc\sqrt{bc}$.

5. Divide $-a^2bc\sqrt{bc}$ by $\sqrt{-a^2bc}$. *Ans.* $+abc\sqrt{-1}$.

6. Divide $-a^2bc\sqrt{bc}$ by $ab\sqrt{-c^2}$. *Ans.* $+a\sqrt{-bc}$.

7. Multiply $-\sqrt{-abc}$ by $-abc\sqrt{-abc}$. *Ans.* $-a^2b^2c^2$.

8. Divide $-a^2b^2c^2$ by $-\sqrt{-abc}$. *Ans.* $-abc\sqrt{-abc}$.

9. Divide $-a^2b^2c^2$ by $-abc\sqrt{-abc}$. *Ans.* $-\sqrt{-abc}$.

10. Add together $a\sqrt{-b^2}$ and $b\sqrt{-a^2}$. *Ans.* $2ab\sqrt{-1}$.

11. Required the square power of $a + b\sqrt{-1}$.
Ans. $a^2 + 2ab\sqrt{-1} - b^2$.

12. Required the square root of $a^2 + 2ab\sqrt{-1} - b^2$.
Ans. $a + b\sqrt{-1}$.

13. Required the third power of $a + b\sqrt{-1}$.
Ans. $a^3 - 3ab^2 + 3a^2b\sqrt{-1} - b^3\sqrt{-1}$.

14. Required the product of $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$.
Ans. a^2+b^2.

15. Required the quotient of a^2+b^2 by $a+b\sqrt{-1}$.
Ans. $a-b\sqrt{-1}$.

16. Required the quotient of a^2+b^2 by $a-b\sqrt{-1}$.
Ans. $a+b\sqrt{-1}$.

17. Required the cube root of $a^3+3a^2b\sqrt{-1}-3ab^2-b^3\sqrt{-1}$.
Ans. $a+b\sqrt{-1}$.

18. Required the fourth power of $a+b\sqrt{-1}$.
Ans. $a^4+4a^3b\sqrt{-1}-6a^2b^2-4ab^3\sqrt{-1}+b^4$.

19. Subtract $2\sqrt[2]{-2}$ from $10\sqrt{-2}$. *Ans.* $8\sqrt{-2}$.

20. Add together $8\sqrt{-2}$ and $2\sqrt[2]{-2}$. *Ans.* $10\sqrt{-2}$.

21. Multiply $\sqrt{-a}+\sqrt{-b}$ by $\sqrt{-a}-\sqrt{-b}$.
Ans. $b-a$.

22. Divide $b-a$ by $\sqrt{-b}+\sqrt{-a}$.
Ans. $-\sqrt{-b}+\sqrt{-a}$.

23. Multiply $\frac{1}{2}+2\sqrt{-a}$ by $\frac{1}{2}-2\sqrt{-a}$. *Ans.* $\frac{1}{4}+4a$.

24. Divide $\frac{1}{4}+4a$ by $\frac{1}{2}+2\sqrt{-a}$. *Ans.* $\frac{1}{2}-2\sqrt{-a}$.

25. Multiply $2\sqrt[4]{-a}$ by $3\sqrt{-a}$. *Ans.* $6\sqrt{-a}\ \sqrt[4]{-a}$.

26. Divide $6\sqrt{-a}\ \sqrt[4]{-a}$ by $2\sqrt[4]{-a}$. *Ans.* $3\sqrt{-a}$.

27. Divide $6\sqrt{-a}\ \sqrt[4]{-a}$ by $3\sqrt{-a}$. *Ans.* $2\sqrt[4]{-a}$.

28. Multiply $\sqrt[4]{-a}+\sqrt[4]{-b}$ by $\sqrt[4]{-a}-\sqrt[4]{-b}$.
Ans. $\sqrt{-a}-\sqrt{-b}$.

29. Multiply $1+b\sqrt{-1}+c\sqrt{-1}$ by $1-b\sqrt{-1}+c\sqrt{-1}$.
Ans. $1+b^2-c^2+2c\sqrt{-1}$.

30. Multiply $1+b\sqrt{-1}+c\sqrt{-1}$ by $1+b\sqrt{-1}-c\sqrt{-1}$.
Ans. $1+c^2-b^2+2b\sqrt{-1}$.

31. Multiply $5\sqrt{-1} + \sqrt{-3}$ by $3\sqrt{-1} - \sqrt{-27}$.

Ans. $-15 - 3\sqrt{3} + 15\sqrt{3} + 9$.

32. Required the second power of $5\sqrt{-1} + \sqrt{-3}$.

Ans. $-25 - 10\sqrt{3} - 3$.

33. Required the square root of $-25 - 10\sqrt{3} - 3$.

Ans. $5\sqrt{-1} - 3$.

34. Divide $6 + \sqrt{-4}$ by $2 + \sqrt{-9}$, and make the denominator of the quotient rational.

Ans. $\dfrac{18 - 14\sqrt{-1}}{13}$.

$$\text{For } \frac{6 + \sqrt{-4}}{2 + \sqrt{-9}} = \frac{(6 + \sqrt{-4})}{(2 + \sqrt{-9})}\,\frac{(2 - \sqrt{-9})}{(2 - \sqrt{-9})} =$$

$$\frac{12 + 2\sqrt{-4} - 6\sqrt{-9} - \sqrt{36}\,\sqrt{-1}\,\sqrt{-1}}{4 + 9} =$$

$$\frac{12 + 4\sqrt{-1} - 18\sqrt{-1} + 6}{13} = \frac{18 - 14\sqrt{-1}}{13}.$$

35. Divide $2 + \sqrt{-2}$ by $2 - \sqrt{-2}$, and make the denominator of the quotient rational.

Ans. $\dfrac{(2 + \sqrt{-2})^2}{6}$ or $\dfrac{2 + 4\sqrt{-2}}{6}$.

36. Divide $1 + \sqrt{-1}$ by $1 - \sqrt{-1}$, and make the denominator of the quotient rational.

Ans. $\sqrt{-1}$.

37. Required the quotient of $6\sqrt{-4}$ divided by $2\sqrt{-9}$.

Ans. 2.

Remarks.

An examination of the foregoing results will show some properties of imaginary quantities.

1. Real, and even rational results, may be obtained by the ordinary algebraic operations upon imaginary quantities.

2. Two monomial imaginaries raised to an even power, will always give a rational result.

3. Two monomial imaginaries multiplied together, or divided, the one by the other, will always give a real result.

4. A monomial imaginary connected by the sign, plus or minus, with one or more rational terms, and then raised to any power, will give at least one imaginary term in the result.

5. Any expression involving one or more monomial imaginaries, may be rendered real by one or more multiplications.

EQUATIONS OF THE SECOND DEGREE.

327. An equation, involving a single unknown quantity, is said to be of the second degree, when the highest exponent of that unknown quantity in any one term is 2. Thus, $x^2 + x = m$, and $x^2 + a = b$, are equations of the second degree with one unknown quantity.

An equation, involving two unknown quantities, is said to be of the second degree when the highest sum of the exponents of the unknown quantities in any term is equal to 2. Thus, $xy = m$, $x^2 + y = n$, and $y^2 + x = p$ are equations of the second degree with two unknown quantities. For they may be written, $x^1y^1 = m$, $x^2y^0 + y = n$, and, $y^2x^0 + x = n$, and, therefore, come within the definition.

We will begin with equations of the second degree, with a single unknown quantity. These are divided into two classes, *complete* and *incomplete*. A complete equation of the second degree, with a single unknown quantity, is one which contains the second and first powers of the unknown quantity. Thus, $x^2 + x = 0$, $dx^2 + x = m$, and $ax^2 + bx = c$, are complete equations of the second degree, with one unknown quantity. There may or may not be other terms, besides those involving the unknown quantity. The first and second powers of the unknown quantity may or may not have coefficients different from unity.

An incomplete equation of the second degree is one containing the second power only of the unknown quantity, and it may or may not involve terms in which the unknown quantity does not enter. The unknown quantity may or may not be affected with a coefficient different from unity. Thus, $ax^2 = x^2$, $x^2 + m = n$, $\frac{x^2}{b} + c = -q$ are incomplete equations of the second degree. Equations of the second degree, with one unknown quantity, are frequently called *quadratic* equations, because the unknown quantity is raised to the second degree, or *squared*.

INCOMPLETE EQUATIONS.

328. We will first examine incomplete equations.

The general form of such equations is $ax^2 - \frac{bx^2}{c} + d = e$. By transposition and reduction, we get $(ca - b)x^2 = ce - cd$, or $x^2 =$

$\frac{c(e-d)}{ca-b}=q$, by substituting for the known terms in the second member a single term, equal to them in value. Hence, every incomplete equation may be reduced to two terms, and the equation be placed under the form of $x^2 = q$. Owing to this circumstance, incomplete equations are sometimes placed in the class of *binomial equations*, or equations involving but two terms.

There is no difficulty in solving the equation $x^2 = q$. The square root of the first member will give us x; but if we extract the square root of the first member, we must extract that of the second also, or the equality will be destroyed. Extracting, then, the root of both members, we have $\pm x = \pm \sqrt{q}$. We have prefixed the double sign to both members, because this equation, when squared, must give back the original equation, $x^2 = q$, from which it was derived, and either $+x$ or $-x$, when squared, will give $+x^2$, and either $+\sqrt{q}$, or $-\sqrt{q}$ will, when squared, give q.

The *value* of the unknown quantity, or *the root* of the equation, as it is generally called, has been defined to be that which, substituted for the unknown quantity, will satisfy the equation, that is, make the two members equal to each other. We see that an incomplete equation of the second degree has two values, numerically equal, but affected with different signs. Either value or root will satisfy the equation, for the result of the substitution of either $\sqrt{q}$, or $-\sqrt{q}$ for x, in the given equation, will be $q = q$. We have prefixed the double sign to both members, but it is usual to prefix it only to the root. In that case, however, we must understand that the sign of x is not necessarily positive, it being affected with the positive sign only, when it corresponds to the positive root. It becomes, then, necessary to have some notation to distinguish the values. This we will do by dashes. The x that corresponds to the positive root we will write x'; and the x that corresponds to the negative root we will write x''. The first is read x prime, and the second, x second. It will be shown that a complete equation has two roots, generally unequal in value. We will also distinguish these roots or values by writing them x', and x''. Some equations of high degrees have three, four, &c., values. These will be written x', x'', x''', x^{iv}, &c., and read x prime, x second, x third, x fourth, and so on.

To solve an incomplete equation, it must first be put under the form of $x^2 = q$. That is, all the known terms must be transferred to the second member, and the coefficient of x^2 must be made plus unity, if not already so, by dividing both members by the coefficient of x^2. After the equation has been put under the form of $x^2 = q$, extract the square root of both members.

EXAMPLES.

1. Solve the equation, $2x^2 + 1 = 4$.

Transposing, and dividing by the coefficient of x^2, we get $x^2 = \frac{3}{2}$. Hence, $x = \pm\sqrt{\frac{3}{2}}$. Then, $x' = +\sqrt{\frac{3}{2}}$, and $x'' = -\sqrt{\frac{3}{2}}$. The solutions may be left thus, or we may extract the indicated root approximatively.

2. Solve the equation, $2x^2 + 4 = 1$.

Reducing, we get $x = \pm\sqrt{-\frac{3}{2}}$. Then, $x' = +\sqrt{-\frac{3}{2}}$, and $x'' = -\sqrt{-\frac{3}{2}}$.

These roots are imaginary. How are we to interpret them? An imagiuary quantity indicates an impossible operation. Ought not the equation which produces it involve an impossibility? In this instance, $2x^2$, an essentially positive quantity, is added to 4, and their sum is required to be less than 4. The condition of the problem is, then, absurd, or impossible, and the result is impossible, as it ought to be. It will be shown more rigorously, hereafter, that an imaginary solution always indicates absurdity in the conditions of the problem. The imaginary values, though they fail to satisfy the conditions of the problem, yet will satisfy the equation of the problem; as they manifestly ought to do, since they have been truly derived from it. Substituting either $+\sqrt{-\frac{3}{2}}$, or $-\sqrt{-\frac{3}{2}}$, for x, in the equation, $2x^2+4=1$, we get $-\frac{6}{2} + 4 = 1$, or $1 = 1$.

329. We observe, then, this remarkable analogy between imaginary values in equations of the second degree, and negative values in equations of the first degree. The values, in both cases, will satisfy the equation of the problem, but will not satisfy the required conditions. There is, however, this difference: to convert a negative solution into a positive one, numerically equal to it, we have only to impose a single condition; but, to convert an imaginary solution into a real solution, precisely equal to it numerically, we have generally to impose two conditions. Thus, the equation which gives the imaginary solutions, expressed in words would read thus: required to find a number, twice the square of which, augmented by 4, will give a sum equal to 1.

The values are $x' = +\sqrt{-\frac{3}{2}}$, and $x'' = -\sqrt{-\frac{3}{2}}$. Now, to get real values numerically equal, and not to make any change upon the arithmetical values of the known quantities in the equation, it must be changed into $2x^2 - 4 = -1$. This equation, expressed in words, would read thus: required to find a number, twice the square of which, diminished by 4, will give a difference equal to -1. The values of the new equation are $x' = +\sqrt{+\frac{3}{2}}$, and $x'' = -\sqrt{+\frac{3}{2}}$. These values are real, and differ from those in the first equation only in the signs of the quantities under the radicals. To get these new values, we made no change, arithmetically, upon the numbers in the first equation, but we made two changes of sign, or, in other words, we imposed two conditions upon the equation of the problem.

3. Solve the equation, $x^2 + b = a$.

Ans. $x' = +\sqrt{a-b}$, $x'' = -\sqrt{a-b}$.

Now, let $a = 4$, and $b = 6$. Then, $x' = +\sqrt{-2}$, and $x'' = -\sqrt{-2}$. To get the real values, $x' = +\sqrt{2}$, and $x'' = -\sqrt{2}$, we must make b and a interchange values. b must be 4, and a, 6; that is, two conditions must be imposed.

4. Solve the equation, $\frac{x^2}{4} - 1 + x^2 - 4 + 5 = 3x^2 - 7$.

Ans. $x' = +2$, $x'' = -2$.

5. Solve the equation, $\frac{x^2}{9} - 1 + x^2 - 9 + 5 = 3x^2 - 22$.

Ans. $x' = +3$, $x'' = -3$.

6. Solve the equation, $\frac{x^2}{16} - 1 + x^2 - 16 + 5 = 3x^2 - 43$.

Ans. $x' = +4$, $x'' = -4$.

7. Solve the equation, $\frac{2}{3}x^2 - \frac{10}{3} + 4x^2 - 20 + 100x^2 - 500 = 995 - 199x^2$.

Ans. $x' = +\sqrt{5}$, $x'' = -\sqrt{5}$.

8. Solve the equation, $\frac{x^2}{2} + \frac{x^2}{4} - \frac{6}{4} - 3 + 7x^2 - 42 + 999x^2 = 5994$.

Ans. $x' = +\sqrt{6}$, $x'' = -\sqrt{6}$.

The preceding examples are simple, and the solutions can be readily obtained. But there are many of a more complicated character, and of course more difficult to solve. No general rules can be given to aid the student; he must exercise his own ingenuity. It is well, however, to make the clearing from fractions the first step in every reduction, and then, if there is a single radical in the equation, it ought to be placed in one member by itself.

EXAMPLES.

1. Solve the equation, $\frac{\sqrt{x^2+144}}{4} + 3 = \frac{x^2}{4}$.

Clearing of fractions, we get $\sqrt{x^2 + 144} + 12 = x^2$.

Placing the radical by itself, we have $\sqrt{x^2 + 144} = x^2 - 12$.

And squaring both members, there results $x^2 + 144 = x^4 - 24x^2 + 144$. Hence, $x^4 = 25x^2$, or $x^2 = 25$. Then, $x' = +5$ and $x'' = -5$.

2. Solve the equation, $\frac{x}{x + \sqrt{x^2 + 44}} = \frac{10}{22}$.

Then, $22x = 10x + 10\sqrt{x^2 + 44}$, or $12x = 10\sqrt{x^2 + 44}$.

Squaring both members, we get $144x^2 = 100x^2 + 4400$, or $44x^2 = 4400$. Hence, $x' = 10$ and $x'' = -10$.

3. Solve the equation, $\frac{\sqrt{x^2 + b^2c^2}}{c} + b = x^2$.

$$Ans.\ x' = \frac{\sqrt{1 + 2bc^2}}{c},\ x'' = -\frac{\sqrt{1 + 2bc^2}}{c}.$$

These results can be verified by substituting either the value of x' or x'' for x in the given equation. Then we will have $\sqrt{\frac{1 + 2bc^2}{c^2} + b^2c^2}$ $= c\left(\frac{1 + 2bc^2}{c^2} - b\right)$, or, by squaring both members, $\frac{1 + 2bc^2}{c^2} + b^2c^2$

$$= c^2\left(\frac{1 + 2bc^2}{c^2} - b\right)^2 = c^2\left(\frac{1 + 4bc^2 + 4b^2c^4}{c^4} - 2b\left(\frac{1 + 2bc^2}{c^2}\right) + b^2\right)$$

$$= c^2\left(\frac{1 + 4bc^2 + 4b^2c^4 - 2bc^2 - 4b^2c^4}{c^4}\right) + b^2c^2 = \frac{1 + 2bc^2}{c^2} + b^2c^2.$$

Hence the equation is satisfied.

4. Solve the equation, $\frac{x}{x + \sqrt{x^2 + a}} = \frac{b}{a}$.

$$Ans.\ x' = \frac{+b}{\sqrt{a - 2b}},\ x'' = \frac{-b}{\sqrt{a - 2b}}.$$

Verify these results. What do they become when $a = 2b$? Why? What, when $a = 0$? Why? What, when $2b > a$? Why?

5. Solve the equation, $\sqrt{p^2 + x^2} + x = mx$.

$$Ans.\ x' = \frac{+p}{\sqrt{m^2 - 2m}},\ x'' = \frac{-p}{\sqrt{m^2 - 2m}}.$$

Verify these results. What do they become when $m = 2$? Why? What, when $m = 0$? Why? What, when $m = 1$? Why? What, when $p = 0$? Why?

When there are two similar radicals, it is best to unite them in the same member.

6. Solve the equation, $\dfrac{b + b\sqrt{1 - x^2}}{b + m\sqrt{1 - x^2}} = \dfrac{a}{n}$.

$$Ans.\ x' = + \frac{\sqrt{a^2 (m^2 - b^2) + 2anb (b - m)}}{nb - am},\ x'' = - \frac{\sqrt{a^2 (m^2 - b^2) + 2anb (b - m)}}{nb - am}$$

For, clearing of fractions, we get $nb + nb\sqrt{1 - x^2} = ab + am \sqrt{1 - x^2}$, or, $(nb - am) \sqrt{1 - x^2} = b (a - n)$, or, $(nb - am)^2 (1 - x^2) = b^2 (a - n)^2$.

Developing, we get $n^2b^2 - 2nbam + a^2m^2 - x^2 (nb - am)^2 = b^2a^2 - 2anb^2 + b^2n^2$. Hence, $a^2 (m^2 - b^2) + 2anb (b - m) = x^2 (nb - am)^2$.

$$\text{Hence,}\quad x' = + \frac{\sqrt{a^2 (m^2 - b^2) + 2anb (b - m)}}{(nb - am)},\ x'' = - \frac{\sqrt{a^2 (m^2 - b^2) + 2anb (b - m)}}{nb - am}.$$

7. Solve the equation, $n\sqrt{p^2 + x^2} + ax = x\sqrt{p^2 + x^2} + an$.

$$Ans.\ x' = + \sqrt{a^2 - p^2},\ \text{and}\ x'' = - \sqrt{a^2 - p^2}.$$

Verify these results. What do they become when $a^2 = p^2$? Why? What, when $p^2 > a^2$? Why?

330. There is another value which does not appear, it is $x = n$. For, by the second principle, Article 321, the rational terms in the two members must be equal. Equating them, we get $ax = an$, or $x = n$. This ought to be so, for the given equation can be put under the form of $(n - x) \sqrt{p^2 + x^2} - a (n - x) = 0$, an equation which can plainly be satisfied when $x = n$. The given equation, previous to the

division by the common factor, $n-x$, was really a cubic equation, and contained three values. We are sometimes enabled to detect a value, as in the above example, by equating the rational factors. When the root of this equation will also satisfy the equation formed by equating the radicals, it is, of course, a value in the given equation.

8. Solve the equation, $p\sqrt{x^2-a^2}+a=n\sqrt{x^2-a^2}+x$.

$$Ans.\ x=a, \text{ or } x=a\,\frac{((p-n)^2+1)}{1-(p-n)^2}.$$

331. There are some expressions in a fractional form, which must be changed into equivalent fractions with rational denominators.

9. Solve the equation, $\dfrac{m+x+\sqrt{2mx+x^2}}{m+x-\sqrt{2mx+x^2}}=n$.

$$Ans.\ x'=m\,\frac{(1+\sqrt{n})^2}{2(2+\sqrt{n})},\quad x''=m\,\frac{(1-\sqrt{n})^2}{2(2-\sqrt{n})}.$$

For, by multiplying numerator and denominator of the fraction in the first member by the numerator, we get $\dfrac{(m+x+\sqrt{2mx}+x^2)^2}{m^2}=n$.

Clear the equation of fractions, and extract the square root of both members. Then, $m+x+\sqrt{2mx+x^2}=\pm m\sqrt{n}$, or $m\,(1\pm\sqrt{n})-x=-\sqrt{2mx+x^2}$. Squaring both members, we get $m^2\,(1\pm\sqrt{n})^2-2mx)1\pm\sqrt{n})+x^2=2mx+x^2$, and, by transposition, $m^2\,(1\pm\sqrt{n})^2=2mx\,(2\pm\sqrt{n})$. Hence, $x=\dfrac{m(1\pm\sqrt{n})^2}{2(2\pm\sqrt{n})}$.

10. Solve the equation, $\dfrac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}-\sqrt{x^2-a^2}}=1$.

Making the denominator rational as before, we have

$$\frac{(\sqrt{x^2+a^2}+\sqrt{x^2-a^2})^2}{2a^2}=1, \text{ or } x^2+a^2+2\sqrt{x^4-a^4}+x^2-a^2=2a^2.$$

Then, $2\sqrt{x^4-a^4}=2a^2-2x^2$, or $\sqrt{x^4-a^4}=x^2-a^2$.

Squaring again, we get $x^4-a^4=x^4-2x^2a^2+a^4$, or $2x^2a^2=2a^4$, or $x^2=a^2$ Hence, $x'=+a$, and $x''=-a$.

It will be seen that these values satisfy the equation. In this example, the second step was not to extract the square root of both members, as in the last example, because the double product of the radicals gave a simple result. When the radicals are of such a form that their double product will not give a simple result, it is best to make the second step the extraction of the square root of both members.

11. Solve the equation, $\dfrac{\sqrt{x^2 + 3b^2} + \sqrt{x^2 - b^2}}{\sqrt{x^2 + 3b^2} - \sqrt{x^2 - b^2}} = 1.$

Ans. $x' = +b,\ x'' = -b.$

12. Solve the equation, $\dfrac{n\sqrt{x^2 + 8m^2} + n\sqrt{x^2 - m^2}}{\sqrt{x^2 + 8m^2} - \sqrt{x^2 - m^2}} = n.$

Ans. $x' = +m,\ x'' = -m.$

Sometimes the first step is squaring both members.

13. Solve the equation, $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{2m}} = \sqrt{\dfrac{m}{n}}.$

Ans. $x' = +\dfrac{m}{n}\sqrt{2an - m^2},\ x'' = -\dfrac{m}{n}\sqrt{2an - m^2}.$

14. Solve the equation, $\sqrt{a + x} + \sqrt{a - x} = \sqrt{2a}.$

Ans. $x' = +a,\ x'' = -a.$

15. Solve the equation, $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{m}} = \dfrac{x}{m}.$

Ans. $x' = +2\sqrt{am - m^2},\ x'' = -2\sqrt{am - m^2}.$

What do these values become when $m = a$? What, when $m > a$?

16. Solve the equation, $\dfrac{2x + 4}{x + 1} = \dfrac{x - 1}{2x - 4}.$

Ans. $x' = +\sqrt{5},\ x'' = -\sqrt{5}.$

17. Solve the equation, $\dfrac{2x + 4}{x + 2} = \dfrac{x - 2}{2x - 4}.$

Ans. $x' = +2,\ x'' = -2.$

What does the second member become when the first of these values is substituted? How do you explain the result.

18. Solve the equation, $\dfrac{x + a}{x - b} = \dfrac{x + b}{x - a}.$

Ans. Both values infinite.

How are these results explained?

19. Solve the equation, $\frac{nx+a}{x-b}=\frac{x+b}{nx-a}$.

$$Ans.\ x' = +\sqrt{\frac{a^2-b^2}{n^2-1}},\ x'' = -\sqrt{\frac{a^2-b^2}{n^2-1}}.$$

What do these values become when $n=1$? Why? What, when $b^2 > a^2$.

20. Solve the equation, $\frac{2x+1}{x-2}=\frac{x+2}{2x-1}$

$$Ans.\ x' = +\sqrt{-1},\ x'' = -\sqrt{-1}.$$

Why are these results imaginary?

21. Solve the equation, $ax+\frac{a}{x}=\frac{bx-a}{x^2-x}-a$.

$$Ans.\ x' = +\sqrt{\frac{b}{a}},\ x'' = -\sqrt{\frac{b}{a}}.$$

22. Solve the equation, $2x+\frac{2}{x}=\frac{8x-2}{x^2-x}-2$.

$$Ans.\ x' = +2,\ x'' = -2.$$

23. Solve the equation, $3x+\frac{3}{x}=\frac{27x-3}{x^2-x}-3$.

$$Ans.\ x' = +3,\ x'' = -3.$$

PROPERTIES OF INCOMPLETE EQUATIONS.

332. 1st. Every incomplete equation of the second degree has two values, and but two, and these values are equal with contrary signs.

For, the general form of the equation is $x^2=q$ or $x^2-q=0$. The first member may be regarded as the difference of two squares, and can, therefore, be placed under the form of $(x-\sqrt{q})(x+\sqrt{q})$. Hence, the equation, $x^2=q$, may be written $(x-\sqrt{q})(x+\sqrt{q})=0$. Now the product of two factors being equal to zero, the equation can be satisfied by placing either factor equal to zero. Therefore, $x-\sqrt{q}=0$, and $x+\sqrt{q}=0$. From which, we get $x=+\sqrt{q}$, and $x=-\sqrt{q}$, or, distinguishing the values by dashes, $x'=+\sqrt{q}$, $x''=-\sqrt{q}$. Since there are but two factors, the equation can be satisfied in but two ways. Hence there are but two values, and we see that these are equal with contrary signs.

By solving directly the equation, $x^2=q$, we would obtain the same results. But the process we have adopted is to be used hereafter for complete equations, and it is well to know that the properties of complete equations are also those of incomplete equations.

2d. Every incomplete equation of the second degree can be decomposed into two binomial factors of the first degree with respect to x; the first factor being the algebraic sum of x and the first value with its sign changed, and the second factor being the arithmetical sum of x and the second value with its sign changed.

The factors have already been obtained, and are $(x - \sqrt{q})$, and $(x + \sqrt{q})$; and we see that these correspond to the enunciation of the second property. The product of these factors is zero. Hence, when we know the two values of an incomplete equation, we can always form the incomplete equation itself which gave those values. We have only to change the signs of the values, and connect them with x. We will then have the binomial factors, and, placing their product equal to zero, we will have the equation required. Thus, form the equation that gives for x the two values, $+2$ and -2. The two factors are $(x-2)$ and $(x+2)$, and the equation is $(x-2)(x+2)=0$, or $x^2-4=0$. The result can be verified by solving the equation, $x^2-4=0$. Or, since in the equation, $(x-2)(x+2)=0$, we have the product of two factors equal to zero, the equation can be satisfied by placing either factor equal to zero. Hence, $x-2=0$ and $x+2=0$. And these equations, when solved, give the values, $+2$ and -2.

EXAMPLES.

1. Find the equation that gives for x the two values, $+a^2$, and $-a^2$.
Ans. $x^2 - a^4 = 0$.

2. Find the equation whose values are $+\sqrt{ab}$, and $-\sqrt{ab}$.
Ans. $x^2 - ab = 0$.

3. Find the equation whose values are $+a-b$, and $+b-a$.
Ans. $x^2 + 2ab - b^2 - a^2 = 0$.

4. Form the equation whose values are $+\sqrt{m-a}$, and $-\sqrt{m-a}$.
Ans. $x^2 + a - m = 0$.

5. Form the equation whose values are $+\sqrt{m-a} - \sqrt{m^2-a^2}$, and $-\sqrt{m-a} + \sqrt{m^2-a^2}$.
Ans. $x^2 - (m-a) - (m^2-a^2) + 2\sqrt{m-a}\sqrt{m^2-a^2} = 0$.
Verify this result by solving the equation.

6. Solve the equation whose values are $a - \sqrt{a^2 - m^2}$, and $-a + \sqrt{a^2 - m^2}$. *Ans.* $x^2 - 2a^2 + m^2 + 2a\sqrt{m^2 - a^2} = 0$.

Verify this result by solving the equation.

7. Form the equation whose values are $+\dfrac{m\sqrt{a}}{\sqrt{m^2 + a^2}}$, and $-\dfrac{m\sqrt{a}}{\sqrt{m^2 + a^2}}$. *Ans.* $(m^2 + n^2)x^2 = am^2$.

8. Form the equation whose values are $\dfrac{m\sqrt{a}}{\sqrt{m^2 - n^2}}$, and $-\dfrac{m\sqrt{a}}{\sqrt{m^2 - n^2}}$.
Ans. $(m^2 - n^2)x^2 = am^2$.

9. Form the equation whose values are $m^2 - n^2$, and $-m^2 + n^2$.
Ans. $x^2 = m^4 - 2m^2n^2 + n^4$.

What do the values become when $m = n$? What, when $m = 0$?

BINOMIAL EQUATIONS.

333. Any binomial equation of the n^{th} degree can be solved as the preceding equations. We have only to extract the n^{th} root of both members.

Let $x^3 = 27$; then, $x = \sqrt[3]{27} = 3$. Let $x^n = q$; then, $x = \sqrt[n]{q}$.

If n be an even number, the radical must have the double sign, and two values, at least, of the equation will be known. If n be an odd number, only one value will be known. It will be demonstrated hereafter, that every equation of the n^{th} degree has n values. The method of finding the other values will then be explained.

Let $x^4 = a^2$; then, $x = \pm\sqrt[4]{a^2} = \pm\sqrt{\sqrt{a^2}} = \pm\sqrt{\pm a}$.

Then, $x' = +\sqrt{+a}$, $x'' = -\sqrt{+a}$, $x''' = +\sqrt{-a}$, $x'''' = -\sqrt{-a}$.

In this example the four values of x have been determined, but it is not often the case that they all can be found.

Let $x^6 = a^2$; then, $x = \pm\sqrt[6]{a^2} = \pm\sqrt[3]{\sqrt{a^2}} = \pm\sqrt[3]{\pm a}$, and four of the six values are known.

Let $x^{12} = a^3$; then, $x = \pm\sqrt[12]{a^3} = \pm\sqrt[4]{\sqrt[3]{a^3}} = \pm\sqrt[4]{a}$, and only two of the twelve values are known.

EXAMPLES.

1. Solve the equation, $\sqrt{x^4 - 6az + 3a^2} = z - 2a$.

Ans. $x' = +\sqrt{z+a}$, $x'' = -\sqrt{z+a}$, $x''' = +\sqrt{-(z+a)}$, $x'''' = -\sqrt{-(z+a)}$.

2. Solve the equation, $x^4 = 256$. *Ans.* $x' = +4$, $x'' = -4$.

3. Solve the equation, $\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2} = m\sqrt{2}$.

Ans. $x' = +\sqrt[4]{a^4 - (m^2 - a^2)^2}$, $x'' = -\sqrt[4]{a^4 - (m^2 - a^2)^2}$.

4. Solve the equation, $x^{10} = 32$.

Ans. $x' = +\sqrt{2}$, $x'' = -\sqrt{2}$,

5. Solve the equation, $\dfrac{\sqrt{p^2 + x^2} + \sqrt{p^2 - x^2}}{\sqrt{m}} = p\sqrt{2}$.

Ans. $x' = +p\sqrt[4]{m(2-m)}$, $x'' = -p\sqrt[4]{m(2-m)}$.

6, Solve the equation, $x^4 = 1$.

Ans. $x' = +1$, $x'' = -1$, $x''' \pm +\sqrt{-1}$, $x'''' = -\sqrt{-1}$.

Verify the preceding results by substituting them in the given equations.

GENERAL PROBLEMS IN BINOMIAL EQUATIONS.

334. 1. The sum of two numbers is a, and the ratio of their squares m: what are the numbers?

Ans. $x' = \dfrac{+a\sqrt{m}}{1+\sqrt{m}}$, $x'' = \dfrac{-a\sqrt{m}}{1-\sqrt{m}}$

For, let $x =$ one number; then, $a - x =$ the other, and, by the conditions, $\dfrac{x^2}{(a-x)^2} = m$. Now, extract the square root of both members, then, $\dfrac{x}{a-x} = \pm\sqrt{m}$. Hence, $\dfrac{x'}{a-x'} = +\sqrt{m}$, and $\dfrac{x''}{a-x''} = -\sqrt{n}$. From the first equation, $x' = +a\sqrt{m} - x'\sqrt{m}$, or

$x'(1+\sqrt{m}) = a\sqrt{m}$, or $x' = \frac{+a\sqrt{m}}{1+\sqrt{m}}$. From the second equation, we get $x'' = -a\sqrt{m} + x''\sqrt{m}$, or $x'' = \frac{-a\sqrt{m}}{1-\sqrt{m}}$. Now, if $m = 1$, the second value is infinite. How is this result to be interpreted? By going back to the equation of the problem, we see that it becomes, under this hypothesis, $\frac{x^2}{(a-x)^2} = 1$; an equation which can only be true when $a = 0$, or when $x = \frac{a}{2}$. The first supposition contradicts the enunciation, the second is the value found for x'. The second value can only exist when there is a contradiction to the statement: it ought, then, to appear under the symbol of absurdity. We may explain the result otherwise; thus: when $m = 1$, the equation in x'' becomes $x'' - x'' = -a$, or $\frac{-x''+x''}{x''} = +\frac{a}{x''}$, or $0 = +\frac{a}{x''}$. This equation is plainly absurd for any finite value of x''.

The value, $x'' = -\frac{a}{0}$, satisfies the equation of the problem, as it manifestly ought to do. We have $\frac{\frac{a^2}{0^2}}{\left(a-\frac{a}{0}\right)^2} = 1$, or $\frac{\frac{a^2}{0^2}}{\frac{(0a-a)^2}{0^2}} = 1$, or, $\frac{a^2}{0^2} = \frac{(0a-a)^2}{0^2}$, or $0^2a^2 = 0^2(0a-a)^2$, or $0 = 0$.

2. The sum of two numbers is a, and the ratio of their fourth powers is m. What are the numbers?

Ans. $x' = +\frac{a\sqrt[4]{m}}{1+\sqrt[4]{m}}$, $x'' = -\frac{a\sqrt[4]{m}}{1-\sqrt[4]{m}}$.

3. Two numbers are to each other as m to n, m being greater than n, and the ratio of their squares is equal to a^2 divided by the square of the greater. What are the numbers?

Ans. First, $\pm\frac{an}{m}$, Second, $\pm\frac{an^2}{m^2}$.

4. Two numbers are to each others as m to n, and the sum of their fourth powers is equal to a^4. What are the numbers?

Ans. First, $\pm\frac{am}{\sqrt[4]{n^4+m^4}}$, Second, $\pm\frac{an}{\sqrt[4]{n^4+m^4}}$.

5. Two numbers are to each other as m to n, and the difference of their fourth powers is a^4. What are the numbers?

Ans. First, $\pm \frac{am}{\sqrt[4]{m^4-n^4}}$, Second, $\pm \frac{an}{\sqrt[4]{m^4-n^4}}$.

What do these values become when $m = n$? Why? What when $m = 0$? Why?

6. The cube root of twice the square of a number is 2. What is the number?

Ans. Either $+ 2$, or $- 2$.

7. The cube root of a times the square of a number is b. What is the number?

Ans. $x' = + \sqrt{\frac{b^3}{a}}$, $x'' = - \sqrt{\frac{b^3}{a}}$.

What do these values become when $a = 0$? What when $b = 0$?

8. The square of a number multiplied by the first power of the same number is equal to 64. What is the number? *Ans.* 4.

9. A man put out a certain sum of money at 6 per cent. interest. The product of the interest upon the money for 6 months by the interest for 4 months was 600 dollars. What was the sum at interest?

Ans. \$1000.

10. A man put out a certain sum of money at 6 per cent. interest. The product of the interests upon it for 3, 6 and 9 months was \$20¼. What was the sum? *Ans.* \$100.

11. The successive quotients, of a quantity divided first by a, and then by b, will, when multiplied together, give a product equal to mn. What is the quantity?

Ans. $x' = + \sqrt{abmn}$, $x'' = - \sqrt{abmn}$.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

335. The most general form of such equations is, $-\frac{ax^2}{b} + \frac{cx}{m} - n = r$; clearing of fractions, we get, $- amx^2 + bcx - bmn = bmr$. Dividing by the coefficient of x^2, $- am$, we have $x^2 - \frac{bcx}{am} + \frac{bn}{a} = - \frac{br}{a}$; and by transposition, $x^2 - \frac{bcx}{am} = - \frac{b(n+r)}{a}$. Now, the second member

may be plainly represented by a single letter, $-q$, and the coefficient of the second may be represented by $-p$. Hence, the given equation assumes the form of $x^2 - px = -q$; in which the highest term of the unknown quantity has a coefficient plus unity. Had the coefficient, $-\frac{a}{b}$ of x^2 in the original equation been affected with the positive sign, it would have reduced to the form of $x^2 + px = q$. Had the coefficient of x^2 been positive, and that of x negative, the equation would have assumed the form of $x^2 - px = q$. Had the last conditions been fulfilled, and the prevailing sign in the second member at the same time negative, the equation would have assumed the form of $x^2 - px = -q$.

And, since every change that may be made upon the signs of the coefficients of x^2 and x, and upon the signs of the known terms, will eventually lead us to one of the preceding forms, we conclude that every complete equation of the second degree may be placed under one of the following forms:

$x^2 + px = q$, First form,

$x^2 - px = +q$, Second form,

$x^2 + px = -q$, Third form,

$x^2 - px = -q$, Fourth form.

When a complete equation is given to be solved, it must first be placed under one of these forms. To do this, clear it of fractions, if it contain any, and then make the coefficient of the first term *plus unity*, if not already so, by dividing by this coefficient. The resulting equation will then be under one of the required forms. The form that the resulting equation assumes will, of course, depend upon the prevailing signs in the given equation.

Reduce the equation, $\frac{x^2}{4} - \frac{1x}{2} + 3 = -\frac{6}{4}$, to one of the four forms.

Clearing of fractions, we have $x^2 - 2x + 12 = -6$, or $x^2 - 2x = -18$. Hence, the equation has assumed the fourth form.

Reduce the equation, $\frac{x^2}{4} - \frac{1x}{2} + 3 = -\frac{6}{5}$ to an equivalent equation, which will appear under one of the four forms.

Multiplying both members by 20, the least common multiple of the denominators, we will get $5x^2 - 10x + 60 = -24$. Now, divide by $+5$, to make the coefficient of x^2 plus unity. Then, $x^2 - 2x + 12 = -\frac{24}{5}$ or $-4\frac{4}{5}$.

We see in this particular case, that we might have obtained the same result by multiplying the original equation by 4. This would have made the coefficient of the first term unity, which is the main point to be attended to; then, by transposition, the equation would have become $x^2 - 2x = -16\frac{4}{5}$. And we see that the equivalent equation is of the fourth form. It frequently happens that it is impossible to make all the terms entire, and the coefficient of the first term at the same time plus unity. But, since this coefficient must always be plus unity, we derive the following rule for reducing any equation to one of the proposed form.

RULE.

Multiply both members of the equation by the reciprocal of the coefficient of the first term, and then transpose all the known terms to the second member.

The multiplier in the equation, $\frac{3}{5}x^2 - 2x = 4$ is, $+\frac{5}{3}$. The multiplier in the equation, $-\frac{3}{5}x^2 - 2x = 4$ is, $-\frac{5}{3}$.

The reason of the rule is obvious, and needs no explanation.

EXAMPLES.

1. Reduce the equation, $\frac{2}{3}x^2 - \frac{1}{3}x + 6 = 4$, to one of the four forms.

$$Ans.\ x^2 - \frac{x}{2} = -3.$$

2. Reduce the equation, $-\frac{2}{3}x^2 - \frac{1}{3}x + 6 = +4$, to one of the four forms.

$$Ans.\ x^2 + \frac{x}{2} = +3.$$

3. Reduce the equation, $\frac{1}{25}x^2 + \frac{1}{5}x = 1$, to one of the four forms.

$$Ans.\ x^2 + 5x = 25.$$

4. Reduce the equation, $\frac{x^2}{25} - \frac{x}{5} = 1$, to one of the four forms.

$$Ans.\ x^2 - 5x = 25.$$

5. Reduce the equation, $-\frac{2}{3}x^2 - \frac{1}{3}x + 6 = +4$, to one of the four forms.

$$Ans.\ x^2 + \frac{x}{2} = 3.$$

336. If the first member of the equation put under one of the four forms is a perfect square, the solution can be as readily effected as in the case of an incomplete equation. For, we will only have to extract

the square root of both members, and then transfer the known term or terms in the first member to the second member, and then the solution will be complete. Suppose that the equation is $x^2 + 2ax + a^2 = b$. Extract the root of both members, then $x + a = \pm \sqrt{b}$. Hence, $x' = -a + \sqrt{b}$, and $x'' = -a - \sqrt{b}$. We see that the equation has two values, and that these are not numerically equal, as in the case of incomplete equations.

Now if, by any artifice, we can make the first member a complete square, it is plain that there will be no difficulty in the solution of a complete equation of the second degree. Let us then assume the equation, $x^2 + px = q$, and examine what modification the equation must undergo, in order that its first member may be made a perfect square. The square of a binomial is composed of the square of the first term, plus the double product of the first by the second, plus the square of the second term. The square of a binomial is, therefore, a trinomial. $x^2 + px$ must then be made a trinomial by the addition of some quantity before the first member will become a perfect square. What is the quantity to be added? Take the expression, $(x + a)^2 = x^2 + 2ax + a^2$. We see that the third term of the trinomial is the square of the second term of the binomial in the first member; and we see that $2ax$, the second term of the trinomial, when divided by $2x$, twice the first term of the binomial, gives a quotient, a, which is the second term of the binomial. This quotient, squared, is the third term of the trinomial. Now, suppose we only knew the first two terms of the trinomial, x^2 and $2ax$, and wished to ascertain what was the binomial, which, when squared, would give these for the first two terms of its square. We would know that the first term of the binomial must be x; and, since $2ax$, the second term of the trinomial is twice this first term by the second term, it is plain that the second term can be found by dividing by $2x$, twice the first term. Having found a, the second term of the binomial, we have only to square it, and the third term of the trinomial will be known.

Apply these principles to the expression, $x^2 + px$, regarded as the first two terms of a trinomial that is a complete square. It is plain that x is the first term of the binomial, the first two terms of whose square are $x^2 + px$. The second term of the binomial can be found by dividing px by $2x$, twice the first term: the quotient, $\frac{p}{2}$, is the second term of the binomial, and its square, $\frac{p^2}{4}$, is the required third term of the trimonial. If then $\frac{p^2}{4}$ be added to $x^2 + px$, the first member will be a perfect square. But, if we add $\frac{p^2}{4}$ to the first member, we must also add it to the second, else the equality of the two members will be destroyed. Hence, the equation, $x^2 + px = q$, can be changed into the equivalent equation, $x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}$. in which the first member is a perfect square.

Now, extract the square root of both members,

$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} + q}$. Hence, $x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}$, and $x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}$.

Either of these values when substituted for x will satisfy the equation. The first, when substituted, will give $\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}\right)^2 + p\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}\right) = q$, or $\frac{p^2}{4} - p\sqrt{\frac{p^2}{4} + q} + \frac{p^2}{4} + q - \frac{p^2}{2} + p\sqrt{\frac{p^2}{4} + q} = q$.

Reducing the first member we have, $q = q$. Hence, the first value satisfies the equation. The second, when substituted, gives $\left(-\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}\right)^2 + p\left(-\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}\right) = q$, or $\frac{p^2}{4} + p\sqrt{\frac{p^2}{4} + q} + \frac{p^2}{4} + q - \frac{p^2}{2} - p\sqrt{\frac{p^2}{4} + q} = q$, or $q = q$.

Hence, the second value also satisfies the equation.

We have taken an equation under the first form, but it is obvious that the principles deduced are applicable to equations under either of the three other forms. Hence, for the solution of a complete equation of the second degree, we have this general

RULE.

Reduce the equation to one of the four forms, by multiplying both members by the reciprocal of the coefficient of the highest power of the unknown quantity, and by transposing all the known terms to the second *member. Next, add the square of half the coefficient of the first power of the unknown quantity to both members, and then extract the square root of both members.*

Finally, separate the unknown quantity from the known quantities by leaving it alone in the first member.

A common error with beginners is, to complete the square without observing whether the coefficient of the first term is plus unity. But it will be seen that the rule requires the first step to be the making of this coefficient plus unity, if not already so.

EXAMPLES.

1. Solve the equation, $-x^2 + 16x = 28$.
   Ans. $x' = +2,\ x'' = +14$.

2. Solve the equation, $x^2 - 4x = -4$.
   Ans. $x' = +2,\ x'' = +2$.

3. Solve the equation, $x^2 + 4x = -4$.
   Ans. $x' = -2,\ x'' = -2$.

It is generally best to reduce the terms under the radical to a common denominator.

Take the equation, $-ax^2 - cx = -m$. This, when solved, gives $x = -\frac{c}{2a} \pm \sqrt{\frac{m}{a} + \frac{c^2}{4a^2}}$. The two terms under the radical can be reduced to a common denominator by multiplying the numerator and denominator of the first by $4a$. If m had been divisible by a, then a would not appear in the denominator of the first term, and the multiplier then would have been $4a^2$. The single term, into which all the known quantities in the second member have been collected after the coefficient of x^2 has been made plus unity, is called the *absolute term*. In the preceding equation, $\frac{m}{a}$ is the absolute term. Then, to reduce the terms under the radical to the same denominator, we multiply numerator and denominator of the absolute term by either 4 times the coefficient of the second power of x, or 4 times the square of this coefficient. This rule is only applicable when the reduced term in the second member was entire, previous to making the coefficient of x^2 plus unity.

4. Solve the equation, $2x^2 - 3x = 5$.

Ans. $x' = +2\frac{1}{2}$, $x'' = -1$.

For, solving, we get, $x = \frac{3}{4} \pm \sqrt{\frac{5}{2} + \frac{9}{16}} = \frac{3}{4} \pm \sqrt{\frac{40}{16} + \frac{9}{16}} = \frac{3}{4} \pm \frac{7}{4} = +2\frac{1}{2}$, or -1.

5. Solve the equation, $2x^2 - 3x = 2$.

Ans. $x' = +2$, $x'' = -\frac{1}{2}$.

For, solving, we get, $x = \frac{3}{4} \pm \sqrt{1 + \frac{9}{16}} = \frac{3}{4} \pm \sqrt{\frac{16+9}{16}} = \frac{3}{4} \pm \frac{5}{4} = +2$, or $-\frac{1}{2}$.

6. Solve the equation, $\frac{x^2}{m^2} + \frac{bx}{c} = \frac{a^2}{m^2} + \frac{ba}{c}$.

Ans. $x' = +a$, $x'' = -a - \frac{bm^2}{c}$.

337. These few examples are given to show how complete equations can be solved. But the solutions can be better understood when some of the properties of these equations are known.

First Property.

Every complete equation of the second degree has two values, and but two, for the unknown quantity.

For, resume the equation, $x^2 + px = q$. Complete the square of the first member, and we have $x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}$. Now, the first member being a square, the second member may be represented by a square also. Hence, $x^2 + px + \frac{p^2}{4} = m^2$, or $(x + p)^2 - m^2 = 0$. Now, the first member being the difference of two squares, may be resolved by the principle of the sum and difference into two factors. Then the equation, $(x + p)^2 - m^2 = 0$, will be changed into the equivalent equation, $(x + p + m)(x + p - m) = 0$. And since, when the product of two factors is equal to zero, the equation can be satisfied by placing either factor equal to zero, we have, $x + p + m = 0$, and $x + p - m = 0$. From which we get $x = -p - m$, and $x = -p + m$; or, distinguishing the values by dashes, $x' = -p - m$, and $x'' = -p + m$. Now, if we replaced m by its value, we would have the solutions previously obtained. We see that there are two values numerically unequal, and, since the equation, $(x+p+m)(x+p-m)=0$, contains but two factors, it can be satisfied in two ways, and only in two ways. Hence, there are two, and only two, values for the unknown quantity.

Second Property.

338. The first member of every equation of the second degree can be decomposed into two factors of the first degree with respect to x, the first factor being the algebraic sum of x and the first value of x with its sign changed; and the second factor being the algebraic sum of x and the second value with its sign changed. The second member of the equation after this decomposition will be zero.

The factors have already been obtained, and are $(x+p+m)$ and $(x+p-m)$. By comparing these factors with x' and x'', we see that $+p+m$ is equal to $-x'$, and that $+p-m$ is equal to $-x''$, and these factors then can be changed into the equivalent, $(x-x')$ and $(x-x'')$; and the equation, $(x+p+m)(x+p-m)=0$, can be changed into $(x-x')(x-x'')=0$.

339. This important property enables us to find the equation whose values are known. We have only to change the sign of the first value, and prefix $+x$ to it, and the first factor will be known; and then to change the sign of the second value, and prefix $+x$ to it, and the second factor will be known. The product of these factors placed equal to zero is the equation required.

Find the equation, whose values are $+2$ and -3. The first factor must be $(x-2)$; the second factor must be $x+3$. Hence, $(x-2)(x+3)=0$, is the equation required. Expanding the first member, we have $x^2+x-6=0$, or $x^2+x=6$. Completing the square, and solving, we get $x=-\frac{1}{2}\pm\sqrt{6+\frac{1}{4}}=-\frac{1}{2}\pm\sqrt{\frac{25}{4}}=-\frac{1}{2}\pm\frac{5}{2}=+\frac{4}{2}$, or $+2$, or $-\frac{6}{2}=-3$. Hence, the process is verified. But the verification might have been made more readily, thus: the equation $(x-2)(x+3)=0$, can be satisfied by placing either factor equal to zero. Hence, $x-2=0$, and $x+3=0$. These equations give the preceding values, $+2$ and -3.

A few examples will make the beginner more familiar with the second property.

EXAMPLES.

1. Form the equation whose values are both -2.
Ans. $(x+2)(x+2)=0$, or $x^2+4x+4=0$.

2. Form the equation whose values are $a+\sqrt{b}$, and $a-\sqrt{b}$.
Ans. $(x-a-\sqrt{b})(x-a+\sqrt{b})=0$, or $x^2-2ax=b-a^2$.

3. Form the equation whose values are $-2m+\sqrt{n+4m^2}$, and $-2m-\sqrt{n+4m^2}$. *Ans.* $x^2+4mx=n$.

4. Form the equation whose values are $a-n+\sqrt{m-2an+n^2}$, and $a-n-\sqrt{m-2an+n^2}$.
Ans. $x^2-2(a-n)x=m-a^2$.

Verify these results by solving the equations.

340. The second property of complete equations sometimes enables us to solve an equation very readily. For, whenever the factors of an equation are apparent, we need not solve the equation itself, but only those factors separately placed equal to zero. Thus, take the equation $x^2+bx=ax$. By transposition we get, $x^2+bx-ax=0$. We see that x is a common factor to the first member, and the equation may be written $x(x+b-a)=0$. And since, when the product of two factors is equal to zero, the equation can be satisfied by placing either equal to zero, we have $x=0$, and $x+b-a=0$. Hence, the two values are 0, and $a-b$. *The general equation,* $x^2+bx-ax=0$, *shows, furthermore, that when the unknown quantity enters into all the terms of the first member of an equation, whose second member is zero, one value of the unknown quantity must always be zero.*

Take, as a further illustration of the use of the second property, the equation, $ax^2-bx^2+ax-bx=bx-ax$. This equation can be written $(a-b)x^2+2(a-b)x=0$, or $x((a-b)x+2(a-b))=0$, or $x(a-b)(x+2)=0$, or $x(a-b)=0$, and $x+2=0$. Hence, $x'=0$, and $x''=-2$.

Again, $x^2-ax+x-a=0$, may be written $x(x-a)+x-a=0$, or $(x-a)(x+1)=0$. Hence, $x'=a$, and $x''=-1$.

We have seen that the first two properties of complete equations are also properties of incomplete equations. The remaining properties are also, as we shall see, common to both classes of equations.

Third Property.

341. The algebraic sum of the values of every complete equation of the second degree is equal to the coefficient of the first power of the unknown quantity, with its sign changed.

For the equation, $x^2 + px = q$, when solved, gave the two values,

$$x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}$$

$$x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}.$$

Adding these equations, member by member, we get $x' + x'' = -p$, as enunciated.

The third property of complete equations is also a property of incomplete equations. For, an incomplete equation of the form, $x^2 = q$, may be written, $x^2 + 0x = q$. The two values of this equation, solved either as a complete or as an incomplete equation, are, $x' = +\sqrt{q}$, and $x'' = -\sqrt{q}$. The sum of these values, $x' + x''$, is plainly zero, which is the coefficient of the first power of x. An incomplete equation then may be regarded as a complete equation, whose two values are numerically equal, but affected with contrary signs.

Fourth Property.

342. The product of the values in every complete equation of the form, $x^2 + px = q$, is equal to the second member or absolute term with its sign changed.

For these values are $x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}$.

and $x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}$.

and their product, $x'x'' = +\frac{p^2}{4} - \left(\frac{p^2}{4} + q\right) = -q$.

This property belongs also to incomplete equations, for the two values of the equation, $x^2 = q$, are $x' = +\sqrt{q}$ and $x'' = -\sqrt{q}$, and their product, $x'x'' = -q$.

We will see, hereafter, that the third and fourth properties belong to equations of every degree, and that the first and second, with some modifications, are also properties of all equations.

EXAMPLES.

1. What is the product, and what the sum, of the values in the equation, $x^2 + 4x = -4$. *Ans.* $x'x'' = +4$, $x' + x'' = -4$.

2. What the product, and sum, in the equation, $x^2 + x = 1$.
Ans. $x'x'' = -1$, $x' + x'' = -1$.

3. What the product, and sum, in the equation, $x^2 - px = -q$.
Ans. $x'x'' = +q$, $x' + x'' = +p$.

Fifth Property.

343. The value of x, in every complete equation of the second degree, is half the coefficient of the first power of x, plus or minus the square root of the square of half this coefficient increased by the absolute term.

For the equation, $x^2 + px = q$, when solved, gives $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$, which agrees with the statement.

It is obvious that this is also a property of incomplete equations, since, for such equations, $p = 0$.

We have shown the foregoing properties by operating upon an equation of the first form, but, since the demonstrations have in no way been dependent upon the signs of p and q, it is obvious that equations under the three other forms enjoy the same properties.

The fifth property enables us to solve an equation directly, without completing the square in the first member; but it is well to require beginners to complete the square, until familiar with the principles.

GENERAL EXAMPLES.

1. Solve the equation, $ax^2 + cx = ma$.

$$Ans.\ x' = -\frac{c}{2a} + \sqrt{m + \frac{c^2}{4a^2}},\ x'' = -\frac{c}{2a} - \sqrt{m + \frac{c^2}{4a2}}.$$

$$\text{or, } x' = -\frac{c + \sqrt{4a^2m + c^2}}{2a},\ x'' = -\frac{c - \sqrt{4a^2m + c^2}}{2a}.$$

2. Solve the equation, $7x^2 - 14x = +280$.

$$Ans.\ x' = 1 + \sqrt{41},\ x'' = 1 - \sqrt{41}.$$

3. Solve the equation, $nx^2 - mx = nm$.

$$Ans.\ x' = +\frac{m + \sqrt{m^2 + 4n^2m}}{2n},\ x'' = \frac{m\sqrt{m^2 + 4n^2m}}{2n}.$$

4. Solve the equation, $ax^2 - bx = mx$.

$$Ans.\ x' = 0,\ x'' = \frac{m + b}{a}.$$

5. Solve the equation, $4x^2 - 5x = 7x$.

$$Ans.\ x' = 0,\ x'' = 3.$$

6. Solve the equation, $nx^2 - mx = 5x$.

$$Ans.\ x' = 0,\ x'' = \frac{5 + m}{n}.$$

How might the answers to 5 and 6 be deduced directly from the answer to 4.

7. Solve the equation, $-x^2 - px = +q$.

$$Ans.\ x' = -\frac{p}{2} + \sqrt{-q + \frac{p^2}{4}},\ x'' = -\frac{p}{2} - \sqrt{-q + \frac{p^2}{4}}.$$

8. Solve the equation, $x^2 + \frac{5}{2}x - \frac{3}{2} = 0$.

$$Ans.\ x' = +\tfrac{1}{2},\ x'' = -3.$$

9. Solve the equation, $4x^2 + 10x - 6 = 0$.

$$Ans.\ x' = +\tfrac{1}{2},\ x'' = -3.$$

10. Solve the equation, $x^2 - \frac{5}{2}x - \frac{3}{2} = 0$.

$$Ans.\ x' = +3,\ x'' = -\tfrac{1}{2}.$$

11. Solve the equation, $x^2 + x + \frac{3}{16} = 0$.

Ans. $x' = -\frac{1}{4}$, $x'' = -\frac{3}{4}$.

12. Solve the equation, $x^2 - x + \frac{3}{16} = 0$.

Ans. $x' = +\frac{3}{4}$, $x'' = +\frac{1}{4}$.

13. Solve the equation, $16x^2 - 16x + 3 = 0$.

Ans. $x' = +\frac{3}{4}$, $x'' = +\frac{1}{4}$.

14. Solve the equation, $x^2 - ax + ab = +bx$.

Ans. $x' = +a$, $x'' = +b$.

15. Solve the equation, $x^2 + ax + ab = -bx$.

Ans. $x' = -a$, $x'' = -b$.

16. Solve the equation, $mx^2 + \frac{mbx}{a} - m = \frac{max}{b}$.

Ans. $x' = \frac{a}{b}$, $x'' = -\frac{b}{a}$.

17. Solve the equation, $mx^2 - \frac{mbx}{a} - m = -\frac{max}{b}$.

Ans. $x' = +\frac{b}{a}$, $x'' = -\frac{a}{b}$.

18. Solve the equation, $x^2 - \frac{ax}{b} - \frac{mx}{n} = -cx - \frac{px}{q}$.

Ans. $x' = 0$, $x'' = \frac{a}{b} + \frac{m}{n} - c - \frac{p}{q}$.

19. Solve the equation, $x^2 + \frac{ax}{b} + \frac{mx}{n} = cx + \frac{px}{q}$.

Ans. $x' = 0$, $x'' = c + \frac{p}{q} - \frac{a}{b} - \frac{m}{n}$.

20. Solve the equation, $x^2 + mx + cx + \sqrt{n}\,x + m^2x = 0$.

Ans. $x' = 0$, $x'' = -m - c - \sqrt{n} - m^2$.

21. Solve the equation, $x^2 - mx - cx - \sqrt{n}\,x - m^2x = 0$.

Ans. $x' = 0$, $x'' = m + c + \sqrt{n} + m^2$.

These equations show that, when the sign of the coefficient of x has been changed, and the other terms left unaltered, the values will be numerically the same, but affected with contrary signs. They show also that, when all the terms after x^2 are negative, the second member being zero, the values will be both positive; and when the terms are all positive, the second member being zero, the values will be both negative.

22. Solve the equation, $mx^2 + px = -q$.

$$Ans.\ x' = \frac{-p + \sqrt{p^2 - 4mq}}{2m},\ x'' = \frac{-p - \sqrt{p^2 - 4mq}}{2m}.$$

What will these values become when $p^2 = 4mq$? What, when $p^2 < 4mq$?

23. Solve the equation, $4x^2 + 24x = -36$.

$$Ans.\ x' = -3,\ x'' = -3.$$

24. Solve the equation, $4x^2 - 24x = -36$.

$$Ans.\ x' = +3,\ x'' = +3.$$

25. Solve the equation, $4x^2 - 24x = -40$.

$$Ans.\ x' = +3 + \sqrt{-1},\ x'' = +3 - \sqrt{-1}.$$

The following important consequences flow from the last four examples.

Whenever the square of the coefficient of x is not greater than four times the product of the coefficient of x^2 into the absolute term, the values will be imaginary, if the sign of the absolute term is negative. And, whenever the square of the coefficient of x is equal to this negative product, the two values of x will become identical.

344. Some equations may be treated either as complete or incomplete.

1. Solve the equation, $\dfrac{x^2}{(a - x)^2} = n$.

$$Ans.\ x' = \frac{a\sqrt{n}}{1 + \sqrt{n}},\ x'' = -\frac{a\sqrt{n}}{1 - \sqrt{n}}.$$

For, expanding the denominator and clearing of fractions, we have, $x^2 = n\ (a^2 - 2ax + x^2)$, or $(1 - n)\ x^2 + 2anx = na^2$, or $x^2 + \dfrac{2an}{1 - n} = \dfrac{na^2}{1 - n}$. Hence, $x' = -\dfrac{an}{1 - n} + \sqrt{\dfrac{na^2}{1 - n} + \dfrac{a^2n^2}{(1 - n)^2}} = \dfrac{-an + \sqrt{na^2\,(1 - n) + a^2n^2}}{1 - n} = \dfrac{-an + a\sqrt{n}}{1 - n} = \dfrac{a\sqrt{n}\,(1 - \sqrt{n})}{1 - n}$.

Now, multiply numerator and denominator by $1 + \sqrt{n}$, and we have,

$\frac{a\sqrt{n}\,(1-n)}{(1-n)\,(1+\sqrt{n})} = \frac{a\sqrt{n}}{1+\sqrt{n}}$. In the same way, x'' can be shown equal to $-\frac{a\sqrt{n}}{1-\sqrt{n}}$. These values for x' and x'' are identical with those before obtained, when the equation, $\frac{x^2}{(a-x)^2} = n$, was treated as an incomplete equation.

2. Solve the equation, $\frac{x^2}{(a+x)^2} = m^2$, as a complete equation.

Ans. $x' = \frac{am}{1+m}$, $x'' = \frac{-am}{1+m}$.

3. Solve the equation, $\frac{x^2}{(x-\sqrt{p})^2} = q$, as a complete equation.

Ans. $x' = \frac{\sqrt{pq}}{\sqrt{q}-1}$, $x'' = \frac{\sqrt{pq}}{\sqrt{q}+1}$.

345. In the foregoing examples, the unknown quantity has been freed from radicals. Whenever it is connected with radicals, it must be freed from them by the preceding rules, and then the process will be the same as in the examples already given.

1. Solve the equation, $\sqrt{mx^2+nx} = \sqrt{p}$

Ans. $x' = -\frac{n+\sqrt{n^2+4mp}}{2m}$, $x'' = -\frac{n-\sqrt{n^2+4mp}}{2m}$.

2. Solve the equation, $\sqrt{m^2x^2} + nx = p$, as an equation of the second degree.

Ans. $x' = \frac{p}{m+n}$, $x'' = \frac{-p}{m-n}$.

3. Solve the equation, $\sqrt{mx^2} + nx = p$, as an equation of the second degree.

Ans. $x' = \frac{p}{\sqrt{m}+n}$, $x'' = -\frac{p}{\sqrt{m}-n}$.

4. Solve the equation, $\frac{\sqrt{1+x}}{1+x} + \frac{1+x}{\sqrt{1+x}} = 3\frac{1}{3}$.

Ans. $x' = +8$, $x'' = -\frac{8}{9}$.

5. Solve the equation, $\frac{\sqrt{a+x}}{a+x} + \frac{a+x}{\sqrt{a+x}} = m$.

Ans. $x' = \frac{m^2-2(a+1)+m\sqrt{m^2-4}}{2}$, $x'' = \frac{m^2-2(a+1)-m\sqrt{m^2-4}}{2}$.

What will these values become when $m = 2$? What, when $m < 2$?

6. Solve the equation, $\frac{\sqrt{3+x}}{3+x} + \frac{3+x}{\sqrt{3+x}} = 2.$

Ans. $x' = -2,\ x'' = -2.$

7. Solve the equation, $\frac{\sqrt{1+x}}{1+x} + \frac{1+x}{\sqrt{1+x}} = 1.$

Ans. $x' = \frac{-3+\sqrt{-3}}{2},\ x'' = \frac{-3-\sqrt{-3}}{2}.$

8. Solve the equation, $\frac{x+a+c}{\sqrt{2cx+4ac}} = 1.$

Ans. $x' = -a + \sqrt{2ac - c^2},\ x'' = -a - \sqrt{2ac - c^2}.$

What do these values become when $c^2 = 2ac$? What, when $c > 2a$?

9. Solve the equation, $\frac{x+2+4}{\sqrt{8x+32}} = 1.$

Ans. $x' = -2,\ x'' = -2.$

10. Solve the equation, $\frac{x+2+5}{\sqrt{10x+40}} = 1.$

Ans. $x' = -2 + \sqrt{-5},\ x'' = -2 - \sqrt{-5}.$

11. Solve the equation, $\sqrt{x^2+bx} = \sqrt{ax} + \sqrt{bx}.$

Ans. $x' = 0,\ x'' = a + 2\sqrt{ab}.$

Verify all the preceding results.

DISCUSSION OF COMPLETE EQUATIONS OF THE SECOND DEGREE.

346. The four forms of these equations are

$$x^2 + px = q,$$
$$x^2 - px = q,$$
$$x^2 + px = -q,$$
$$x^2 - px = -q,$$

and the corresponding values.

$$\left| \begin{array}{l} x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}, \\ x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}, \end{array} \right. \quad \text{First form.}$$

$$\left.\begin{aligned} x' &= \frac{p}{2} + \sqrt{\frac{p^2}{4} + q}, \\ x'' &= \frac{p}{2} - \sqrt{\frac{p^2}{4} + q}, \end{aligned}\right\} \quad \text{Second form.}$$

$$\left.\begin{aligned} x' &= -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \\ x'' &= -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}, \end{aligned}\right\} \quad \text{Third form.}$$

$$\left.\begin{aligned} x' &= \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \\ x'' &= \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}, \end{aligned}\right\} \quad \text{Fourth form.}$$

Now, we observe that the first and second forms differ only in the sign of x, and that x' in the first form is the same as x'' in the second, taken with a contrary sign; x'' in the first form is the same as x' in the second, taken with a contrary sign. A similar remark may be made in regard to the values of the third and fourth forms. We conclude, then, that to change the signs of the values, without altering them numerically, we have only to change the sign of the coefficient of the first power of x. Thus, $x^2 + x = 2$, when solved, gives $x' = 1$, $x'' = -2$; and $x^2 - x = 2$, gives $x' = 2$, $x'' = -1$.

In like manner, $x^2 - 5x = 4$, when solved, gives the two values, $x' = +4$, $x'' = +2$; and $x^2 + 5x = -4$, when solved, gives $x' = -1$, $x'' = -4$.

If we proceed to extract the square root of $\frac{p^2}{4} + q$, in the two values of the first form, we will get $\frac{p}{2}$, plus a series of other terms. Hence, the two values will become $x = -\frac{p}{2} + \frac{p}{2}$ + other terms; and $x'' = -\frac{p}{2} - \frac{p}{2}$ — other terms.

Hence, the first value is positive and the second negative, and the second is numerically greater than the first, because the radical is added to the quantity without the radical in the second value, and subtracted from it in the first value.

By extracting the square root of $\frac{p^2}{4} + q$ approximatively in the values of the second form, we would see that the first value was positive and the second negative, and that the second was numerically greater than the first. We have anticipated this, from what has been said before, as to the interchange of position and the change of sign between the values of the first and second form.

In the third form both values are negative, because $-\frac{p}{2}$ is $>$ $\sqrt{\frac{p^2}{4} - q}$. The second value is numerically greater than the first.

In the fourth form both values are positive, and the first is numerically greater than the second. These results might have been anticipated from our knowledge of those in the third form.

347. The discussion of the signs of the values in the four forms may be made otherwise, thus: we know, from the fourth property of equations of the second degree, that the product of the values in the first form must be equal to $-q$. Hence, the values must have contrary signs. And we know, from the third property, that the sum must be equal to $-p$. Hence, the negative value is the greater. For, when we add a negative and positive quantity together, and their sum is negative, we know that the negative quantity is greater than the positive.

For the second form, the product of the values is equal to $-q$. Hence, the values have different signs. Their product is equal to $+p$. Hence, the positive value is greater than the negative.

For the third form, the product of the values is equal to $+q$. Hence, the values are both positive, or both negative. But their sum is equal to $-p$. Hence, they are both negative. We cannot decide from this, however, which is the greater, the first or second value.

For the fourth form, the product of the values is equal to $+q$. Hence, the values must have like signs, and must be both positive, or both negative. But their sum is positive, being equal to $+p$; hence, they are both positive. We cannot decide from this, however, which is the greater, the first or the second value.

IRRATIONAL, IMAGINARY, AND EQUAL VALUES.

348. The values, in all the forms, will be irrational whenever the root of the radical cannot be extracted exactly. In that case, the approximate value of the radical can be determined to within a vulgar or decimal fraction, and then the approximate values of the unknown quantity will be known.

There can be no imaginary values in the first two forms, because the quantities under the radical are affected with positive signs. Whenever, then, the absolute term of an equation is positive, we know, certainly, that there are no imaginary values in the equation.

The third and fourth forms contain imaginary values, only when q is $> \frac{p^2}{4}$. For, in that case only, is the prevailing sign under the radical negative. Whenever, then, an equation is put under the third or fourth form (the unknown quantity being freed from radicals, and the coefficient of x^2 being made plus unity), we can readily tell whether there are imaginary values. We have only to square half the coefficient of the first power of x, and compare the result with the absolute term.

The equations, $x^2 + 10x = -30$, and $x^2 - 10x = -30$, both contain imaginary values, because $(5)^2 < 30$. Imaginary values, being always similar in form, are said to enter the equation *in pairs*.

Since the two values of each of the four forms differ only by the radical, it is obvious that the disappearance of the radical will cause these values to become identical. But, the radical cannot be made to vanish in the first and second forms, because the quantities under the sign are positive. It will vanish, however, in the third and fourth forms, whenever $\frac{p^2}{4} = q$. The values of the third form will then both become $-\frac{p}{2}$, and of the fourth form $+\frac{p}{2}$. The equality of the values, in the third and fourth forms, differs from the equality of the values of incomplete equations. In the latter, it is equality only in a numerical sense; in the third and fourth forms, absolute equality.

If we substitute $\frac{p^2}{4}$ for q in the third and fourth forms, the equations will become $x^2 + px = -\frac{p^2}{4}$, and $x^2 - px = -\frac{p^2}{4}$, or, by transposition, $x^2 + px + \frac{p^2}{4} = 0$, and $x^2 - px + \frac{p^2}{4} = 0$.

The first member of one equation is the square of $(x + \frac{p}{2})$, and of the other $(x - \frac{p}{2})$. Hence, when there are equal values in the third and fourth forms, the first members of those equations will be perfect squares, provided the second members are zero.

The following are illustrations: $x^2 + 6x = -9$, and $x^2 - 6x = -9$; $x^2 + 8x = -16$, and $x^2 - 8x = -16$.

SUPPOSITIONS MADE UPON THE CONSTANTS.

349. Known terms are frequently called constants, and unknown terms, variables. In the equation, $x^2 + px = q$, p and q are the constants, and x the variable.

Let us make the constant $q = o$, in the four forms. By going back to the solved equations, we see that the values will become

$$\left.\begin{aligned} x' &= -\frac{p}{2} + \frac{p}{2} = 0, \\ x'' &= -\frac{p}{2} - \frac{p}{2} = -p, \end{aligned}\right\} \quad \text{First form.}$$

$$\left.\begin{aligned} x' &= \frac{p}{2} + \frac{p}{2} = p, \\ x'' &= \frac{p}{2} - \frac{p}{2} = 0, \end{aligned}\right\} \quad \text{Second form.}$$

$$\left.\begin{aligned} x' &= -\frac{p}{2} + \frac{p}{2} = 0, \\ x'' &= -\frac{p}{2} - \frac{p}{2} = -p, \end{aligned}\right\} \quad \text{Third form.}$$

$$\left.\begin{aligned} x' &= +\frac{p}{2} + \frac{p}{2} = p, \\ x'' &= +\frac{p}{2} - \frac{p}{2} = 0, \end{aligned}\right\} \quad \text{Fourth form.}$$

We see that the first and third forms have the same values, and that the second and fourth forms have the same values. This ought to be so, for the hypothesis, $q = 0$, makes the first and third equations identical, and also makes the second and fourth equations identical.

We have gotten the foregoing series of values by making $q = 0$ in the solved equations. Ought we not to get the same results by operating upon the given equations themselves? When $q = 0$, the first and third forms become $x^2 + px = 0$, or $x(x + p) = 0$; an equation which can be satisfied by placing either factor equal to zero. Hence, $x = 0$, and $x + p = 0$; or, $x' = 0$, and $x'' = -p$; the same as before obtained. The second and fourth forms become, when $q = 0$, $x^2 - px = 0$, or $x(x - p) = 0$. Hence, $x'' = 0$, and $x' = +p$, as before.

When $p = 0$, we have this series of values:

$x' = +\sqrt{q}$, $x'' = -\sqrt{q}$, in the first and third forms.

$x' = +\sqrt{-q}$, $x'' = -\sqrt{-q}$, in the second and fourth forms.

These results can be obtained directly from the equations; for the first and second become $x^2 = q$, and the third and fourth, $x^2 = -q$. These two equations will plainly give the four preceding values.

When $p = 0$, and $q = 0$, the system of values reduces to $x' = +0$, $x'' = +0$, in each of the forms. This ought to be so; for, in that case, the four equations reduce to the same, $x^2 = 0$.

The solutions of the four equations give formulas which can be applied to particular examples. Thus, the values of the first form are $x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}$, and $x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}$. Let it be required, now, to apply these results as formulas to find the values of the equation, $x^2 + x = 2$. Then, $p = 1$, and $q = 2$. Hence, $x' = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2} = -\frac{1}{2} + \frac{3}{2} = 1$. And $x'' = -\frac{1}{2} - \sqrt{\frac{1}{4} + 2} = -\frac{1}{2} - \frac{3}{2} = -2$.

We have used as formulas the values x' and x'', belonging to the first form, because the equation, $x^2 + x = 2$, is of that form. And, of course, we must always apply as formulas the values found in the form to which the given equation belongs.

EXPLANATION OF IMAGINARY VALUES.

350. We have seen that the values of the third and fourth forms become imaginary when q was made greater than $\frac{p^2}{4}$. It remains to explain the cause of the imaginary results, and to ascertain what they mean.

In the third form, the product of the values is equal to $+q$, and their sum equal to $-p$. The values, then, are both negative; and we have previously seen that they were unequal when $\frac{p^2}{4}$ was unequal to q. If they were equal, each must be $-\frac{p}{2}$. Let x represent the excess of the greater over $-\frac{p}{2}$. Then the value which is numerically the greater will be represented by $-\left(\frac{p}{2}+x\right)$, and the smaller numerically will be represented by $-\left(\frac{p}{2}-x\right)$. Their product will be $\frac{p^2}{4}-x^2$. Now, it is evident that this product will be the greatest possible, when $x=0$. This condition makes the two values equal, and makes their greatest product $\frac{p^2}{4}$. But q represents their product, and it, therefore, can never be greater than $\frac{p^2}{4}$. Hence, when we make $\frac{p^2}{4}>q$, we impose an impossible condition. *An imaginary solution, then, indicates an impossible condition.*

Ought not the equation to point out an absurdity when an impossible condition is imposed? When the square is completed in the third form, the equation is $x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q$, or $\left(x+\frac{p}{2}\right)^2=\frac{p^2}{4}-q$. Now, when $q>\frac{p^2}{4}$, the second member is essentially negative, whilst the first, being a square, is essentially positive. We then have a negative quantity equal to a positive quantity, which is absurd. *A careful inspection, then, of the equation which produces imaginary values will show an absurdity.*

We have examined only the third form, but the preceding consequences may be readily deduced by an examination of the fourth form.

The following problem will illustrate more fully the subject of imaginary values.

Required to divide the number 10 into two parts; such, that their product will be equal to 30.

Let x be one of the parts, then $10-x$ will be the other, and, from the conditions we get $x(10-x)=30$, or $-x^2+10x=30$. Multiplying both members by minus unity, we get $x^2-10x=-30$, and, completing the square, $x^2-10x+25=-5$. Hence, $x'=5+\sqrt{-5}$, and $x''=5-\sqrt{-5}$

The values are imaginary, as they ought to be; for the greatest product that we can form of two numbers, whose sum is 10, is 25. Hence, we have imposed an impossible condition, and the imaginary values point out this impossibility. The equation, $x^2 - 10x + 25 = -5$, may be written $(x - 5)^2 = -5$; that is, a positive quantity equal to a negative one, which is absurd. The imaginary values will, however, satisfy the given equation, as they ought to do, since they have been derived from it.

EXPLANATION OF NEGATIVE SOLUTIONS.

351. Required to find a number whose square, augmented by three times the number, and also by 4, shall be equal to 2.

The equation of the problem is $x^2 + 3x + 4 = 2$.

This equation, plainly, cannot be satisfied in an arithmetical sense; for 4 is already greater than 2, and must be still greater when augmented by x^2 and $3x$. Solving the equation, we get $x' = -1$, and $x'' = -2$.

These negative values satisfy the equation, but do not fulfil the conditions of the enunciation. For, x being negative, the equation will become $x^2 - 3x + 4 = 2$. The true enunciation of the problem, then, is: Required to find a number, whose square, diminished by three times the number, and that remainder increased by 4 will give a result equal to 2. The equation of this problem, when solved, will give the two positive values, 1 and 2.

Negative values then, in equations of the second as well as of the first degree, satisfy the equation of the problem, but do not fulfil the enunciated conditions; and, since negative quantities indicate a change of direction or character, those negative values point out the change that must take place in the enunciation, in order that its conditions may be complied with.

GENERAL PROBLEMS INVOLVING COMPLETE AND INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

352. 1. Required to divide the number n into two parts, the product of which shall be equal to m.

Ans. $x' = \frac{1}{2}n + \sqrt{\frac{1}{4}n^2 - m}$, $x'' = \frac{1}{2}n - \sqrt{\frac{1}{4}n^2 - m}$.

What will these values become when $m = \frac{1}{4}n^2$? What, when $m > \frac{1}{4}n^2$? Why?

2. Required to divide the number 10 into two such parts that their product shall be equal to 25. *Ans.* $x' = 5$, $x'' = 5$.

Is the disappearance of the radical always connected with maximum values?

3. Required to divide the number 10 into two such parts that their product shall be equal to 26.

Ans. $x' = 5 + \sqrt{-1}$, $x'' = 5 - \sqrt{-1}$.

Why are these values imaginary? Verify them by substitution in the equation of the problem.

4. The sum of two numbers is a, and the sum of their squares is b, what are the numbers?

$$Ans.\ x' = \frac{a + \sqrt{2b - a^2}}{2},\ x'' = \frac{a - \sqrt{2b - a^2}}{2}.$$

When do these values become imaginary? when equal? What is the least value that the sum of the squares can have?

5. The sum of two numbers is 10, and the sum of their squares is 52, what are the numbers? *Ans.* $x' = 6$, $x'' = 4$.

6. The sum of two numbers is 10, and the sum of their squares is 50, what are the numbers? *Ans.* $x' = 5$, $x'' = 5$.

7. The sum of two numbers is 10, and the sum of their squares is 40, what are the numbers?

Ans. $x' = 5 + \sqrt{-5}$, $x'' = 5 - \sqrt{-5}$.

How may the values, in the last three examples, be deduced directly from the general values in example 4?

8. A Yankee pedlar bought a certain number of clocks, which he sold again for m dollars. His gain per cent. on his investment was expressed by the number of dollars in it. How much did he pay for the clocks?

Ans. $x' = -50 + \sqrt{(25 + m)100}$, $x'' = -50 - \sqrt{(25 + m)100}$.

9. Same problem as the last, except that the pedlar sold his clocks for 375 dollars, instead of m dollars.

Ans. $x' = 150$, $x'' = -250$.

What is the meaning of the negative value?

10. A number of partners in business owe a debt of m dollars, but, in consequence of the failure of one of their number, each of the solvent partners has to pay n dollars more than his proper proportion of the debt. How many partners were there.

$$Ans.\ x' = \frac{1 + \sqrt{\frac{4m}{n} + 1}}{2},\ x'' = \frac{1 - \sqrt{\frac{4m}{n} + 1}}{2}.$$

11. Same problem as last, except that the debt was 728 dollars, and that the increased portion of the solvent partners was $60\frac{2}{3}$ dollars.

Ans. $x' = 4,\ x'' = -3.$

The negative value is readily explained. An examination of the equation of the problem, after $-x$ has taken the place of $+x$, will show that there have been two changes of condition, and the corresponding enunciation will be: "Several partners in trade owed a debt of 728 dollars, and, by the accession of another partner to share the debt with them, their individual liability was diminished by $60\frac{2}{3}$; required the number of partners." These changes in the enunciation give $+3$ for the number of partners, and the result can be verified. For, the share of each in the debt is, $242\frac{2}{3}$ dollars before a new partner was added to the firm, and but 182 dollars afterwards. And, in general, for problems involving equations of the second degree, there must be two changes in the enunciation to convert a negative into a positive solution.

12. The difference between the cube and the square of a number is equal to twice the number. Required the number.

Ans. $x' = +2,\ x'' = -1.$

13. Required to divide a quantity, a, into two such parts, that the greater part shall be a mean proportional between the whole quantity and the smaller part. What are the two parts?

Ans. Greater part, either $\dfrac{-a + a\sqrt{5}}{2}$, or $\dfrac{-a - a\sqrt{5}}{2}$; and lesser part, either $\frac{3}{2}a - \frac{1}{2}a\sqrt{5}$, or $\frac{3}{2}a + \frac{1}{2}a\sqrt{5}$.

Both values will satisfy the enunciation, in one sense; for, $a(\frac{3}{2}a - \frac{1}{2}a\sqrt{5}) = \left(\dfrac{-a + a\sqrt{5}}{2}\right)^2$, and $a(\frac{3}{2}a + \frac{1}{2}a\sqrt{5}) = \left(\dfrac{-a - a\sqrt{5}}{2}\right)^2$.
Because, by performing the indicated operations, we have for the first

value, $\frac{3}{2}a^2 - \frac{1}{2}a^2\sqrt{5} = \frac{a^2}{4} - \frac{a^2}{2}\sqrt{5} + \frac{5a^2}{4} = \frac{3}{2}a^2 - \frac{1}{2}a^2\sqrt{5}$; and, for the second value, $\frac{3}{2}a^2 + \frac{1}{2}a^2\sqrt{5} = \frac{a^2}{4} + \frac{1}{2}a^2\sqrt{5} + \frac{5a^2}{4} = \frac{3}{2}a^2 + \frac{1}{2}a^2\sqrt{5}$.

The second value, however, does not satisfy the conditions of the problem, understood in a literal sense, for the greater part, $-\frac{a}{2} - \frac{a}{2}\sqrt{5}$, is less than the corresponding smaller part, $\frac{3}{2}a + \frac{a}{2}\sqrt{5}$, and this corresponding part is greater than the whole quantity, a. The explanation of the negative solution is simple, when we return to the equation of the problem, $x^2 = a(a - x)$. When x is negative, this equation becomes, $x^2 = a(a + x)$, and the corresponding enunciation ought to be required to find a number, which shall be a mean between the whole quantity, a, and the sum of a and this number.

The given quantity, a, might have been represented by a straight line, B————C————A, and the problem then would have been to find a part, BC, which should be a mean between the whole, BA, and the part, AC, left after BC was taken from it. Now, when the expression for BC became negative in the solution of the problem, it indicated that BC must be laid off on the left of the point B, because this distance was laid off on the right of B, when its expression was positive. The diagram becomes C————B————A, and AC is greater than AB. This agrees with the second value of AC, $\frac{3}{2}a + \frac{a}{2}\sqrt{5}$, which is greater than a or AB. The negative solution here indicates then a change of direction, and the equation being of the second degree, there are two corresponding changes of condition. The first is expressed by the unknown distance being sought upon the prolongation of AB, instead of upon AB itself; the second is expressed by the unknown distance being a mean between AB, and AB *plus* BC, instead of between AB and AB *minus* BC.

The explanation of a negative solution need never be difficult; we have only to change $+x$ into $-x$ in the equation of the problem, and then examine and see what the resulting equation means.

Some of the following problems will involve one of the four methods of elimination. Whenever the equations between which the elimination is to be effected is of a higher degree than the first, the method of elimination by the greatest common divisor ought to be employed.

14. Find two numbers whose sum and product are both equal to a.

$$Ans.\ x' = \frac{a + \sqrt{a^2 - 4a}}{2},\ x'' = \frac{a - \sqrt{a^2 - 4a}}{2},$$

$$y' = \frac{a - \sqrt{a^2 - 4a}}{2},\ y'' = \frac{\sqrt{a^2 - 4a} + a}{2}.$$

When will the two numbers be equal? When imaginary?

15. Find two numbers whose sum and product shall be equal to 4.

Ans. 2 and 2.

16. Find two numbers whose sum and product shall be equal to 2.

Ans. $x' = 1 + \sqrt{-1}$, $x'' = 1 - \sqrt{-1}$, $y' = 1 - \sqrt{-1}$, $y'' = 1 + \sqrt{-1}$.

17. Find two numbers whose sum and product shall be both equal to 5.

$$Ans.\ x' = \frac{5 + \sqrt{5}}{2},\ x'' = \frac{5 - \sqrt{5}}{2},\ y' = \frac{5 - \sqrt{5}}{2},\ y'' = \frac{5 + \sqrt{5}}{2}.$$

Do the values in problems 14, 16 and 17, indicate that there are two distinct sets of numbers, or that the second set is the same as the first, differing only in the position of the numbers.

18. Two capitalists, A and B, invested different sums in trade. A invested half as much as B, and kept his money in trade 6 months, and gained one-twentieth of his investment. B kept double the amount that A had, in trade for 9 months, and gained $100 more than A. Supposing that their respective gains were proportional to their respective capitals and the periods of investment, what were their capitals? *Ans.* A's $1000, B's $2000.

Why was the second value of x rejected? What was the gain per cent of A and B?

19. The difference between two numbers is a, and the difference between their cubes is b; what are the numbers?

$$Ans.\ x' = \frac{a}{2} + \sqrt{\frac{4b - a^3}{12a}},\ x'' = \frac{a}{2} - \sqrt{\frac{4b - a^3}{12a}},$$

$$y' = -\frac{a}{2} + \sqrt{\frac{4b - a^3}{12a}},\ y'' = -\frac{a}{2} - \sqrt{\frac{4b - a^3}{12a}}.$$

When will x'' and y'' be real, but negative?

What do the negative values satisfy? When will the two values of x be equal? How will the two values of y be in that case? When will the solutions be indeterminate? How many conditions must be imposed?

20. The difference of two numbers is 2, and the difference of their cubes 152; what are the numbers?

Ans. First number, + 6, or — 4; second number, + 4, or — 6.
How are the negative values explained?

21. The difference of two numbers is 4, and the difference of their cubes 16; what are the numbers?

Ans. First number, + 2; second number, — 2.

22. The difference of two numbers is 4, and the difference of their cubes 15; what are the numbers?

Ans. $x' = 2 + \sqrt{\frac{-1}{12}}$, $x'' = 2 - \sqrt{\frac{-1}{12}}$, $y' = -2 + \sqrt{\frac{-1}{12}}$, $y' = -2 - \sqrt{\frac{-1}{12}}$.

23. The difference of two numbers is zero, and the difference of their cubes zero; what are the numbers?

Ans. x' and x'', both $= \frac{0}{0}$, and y' and y'', also both $= \frac{0}{0}$.

24. The year in which the translation began of what is called King James' Bible, is expressed by four digits. The product of the first, second and fourth, is 42; the fourth is one greater than the second, and the sum and difference of the first and third are both equal to one. Required the year. *Ans.* 1607.

25. The year in which Decatur published his official letter from New London, stating that the traitors of New England burned blue lights on both points of the harbour to give notice to the British of his attempt to go to sea, is expressed by four digits. The sum of the first and fourth is equal to half the second; the first and third are equal to each other; the sum of the first and second is equal to three times the fourth, and the product of the first and second is equal to 8. Required the year. *Ans.* 1813.

26. The year in which the Governors of Massachusetts and Connecticut sent treasonable messages to their respective Legislatures, is expressed by four digits. The square root of the sum of the first and

second is equal to 3; the square root of the product of the second and fourth is equal to 4; the first is equal to the third, and is one-half of the fourth. Required the year. *Ans.* 1812.

27. A gentleman puts out a certain capital at an interest of 5 per cent.; the product of the interest for 12 months by the interest for one month, is just one fourth of the principal. What is the capital?

Ans. $x' = 0$, $x'' = \$1200$.

28. Some of the New England States were fully, and some partially, represented in the Hartford Convention, which, in the year 1814, gave aid and comfort to the British during the progress of the war. If 4 be added to the number of States fully and partially represented, and the square root of the sum be taken, the result will be the number of States fully represented; but if 11 be added to the sum of the States fully and partially represented, and the square root of the sum be taken, the result will be equal to the square root of 8 times the number of States partially represented. Required the number of States fully and partially represented.

Ans. Three fully represented; two partially represented.

29. The sum of two numbers is a, the sum of their squares b, and the sum of their cubes c. What are the numbers?

Ans. $x' = \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + \frac{4(c-ab)}{a}}$, $x'' = \frac{a}{2} - \frac{1}{2}\sqrt{a^2 + \frac{4(c-ab)}{a}}$,
$y' = \frac{a}{2} - \frac{1}{2}\sqrt{a^2 + \frac{4(c-ab)}{a}}$, and $y'' = \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + \frac{(4c-ab)}{a}}$.

Have we two distinct sets of values, or but one set, with an interchange of position? When will the values of x and y be equal? When imaginary? When negative?

30. The sum of two numbers is 7, the sum of their squares 25, and the sum of their cubes 91. What are the numbers?

Ans. $x' = 4$, $x'' = 3$; $y' = 3$, $y'' = 4$.

31. The sum of two numbers is 6, the sum of their squares is 18, and the sum of their cubes 54. What are the numbers?

Ans. Both 3.

32. The sum of two numbers is 5, the sum of their squares 20, and the sum of their cubes 50. What are the numbers?

Ans. $x' = \frac{5}{2} + \frac{1}{2}\sqrt{-15}$, $x'' = \frac{5}{2} - \frac{1}{2}\sqrt{-15}$; $y' = \frac{5}{2} - \frac{1}{2}\sqrt{-15}$, $y'' = \frac{5}{2} + \frac{1}{2}\sqrt{-15}$.

33. Two travellers started at the same time from two cities, C and W, and travelled toward each other. They found, on meeting, that the traveller from C had travelled 150 miles more than the other traveller, and that by continuing at the same rate, he could reach W in five days; whereas, it would take the traveller from W twenty days from the time of meeting to reach C. Required the distance between W and C, the rate of travel per day of the two travellers, and the time that had elapsed before their meeting.

Ans. Distance between W and C, 450 miles; the rates, 30 and 15 miles per day; the time elapsed before meeting, 10 days.

34. The sum of three numbers is 15, the sum of their squares 93, and the third is half the sum of the first and second. What are the numbers? *Ans.* 8, 2 and 5.

35. A man bought a tract of land for $11 per acre, and sold it again at a less price, his loss per cent. on the sale being expressed by the price per acre which he received. What did he sell the land for?

Ans. 10 dollars per acre.

36. In the year 1637, all the Pequod Indians that survived the slaughter on the Mystic River were either banished from Connecticut, or sold into slavery. The square root of twice the number of survivors is equal to $\frac{1}{10}$th that number. What was the number? *Ans.* 200.

37. A Southern Planter bought m acres of cultivated, and as many of uncultivated, land. He got b more acres of uncultivated than of cultivated land per dollar, and the whole cost of the cultivated exceeded that of the uncultivated by c dollars. How much did he pay per acre for each kind of land?

Ans. $x' = \dfrac{1}{\dfrac{-b}{2} + \frac{1}{2}\sqrt{\dfrac{b(4m+bc)}{c}}}$, $x'' = \dfrac{1}{\dfrac{-b}{2} - \frac{1}{2}\sqrt{\dfrac{b(4m+bc)}{c}}}$, for cultivated land; $y' = \dfrac{1}{\dfrac{b}{2} + \frac{1}{2}\sqrt{\dfrac{b(4m+bc)}{c}}}$, $y'' = \dfrac{1}{\dfrac{b}{2} - \frac{1}{2}\sqrt{\dfrac{b(4m+bc)}{c}}}$, for uncultivated land.

38. A Southern Planter purchased 100 acres of cultivated, and 100 acres of uncultivated land, the former costing him $500 more than the

latter; the smaller cost of the latter resulted from his getting for every dollar $\frac{1}{10}$th of an acre more of the uncultivated land than of the cultivated. What was the cost per acre of the cultivated and uncultivated land.

Ans. Former, \$10 per acre; latter, \$5 per acre.

Verification. 100 acres at \$10 per acre will cost \$1000, and 100 acres at \$5 will cost \$500. A dollar will buy $\frac{1}{10}$th of an acre of the cultivated, and $\frac{1}{5}$th of an acre of the uncultivated land; and the difference between $\frac{1}{5}$th and $\frac{1}{10}$th is $\frac{1}{10}$th. So, a dollar will buy $\frac{1}{10}$th of an acre more of the uncultivated than of the cultivated land.

39. In the year 1853, a number of persons in New England and New York, were sent to lunatic asylums in consequence of the Spiritual Rapping delusion. If 14 be added to the number of those who became insane, and the square root of the sum be taken, the root will be less than the number by 42. Required the number of victims.

Ans. 50.

Why is the second value of x rejected?

40. Two farmers, A and B, invest each a certain amount in the Central Railroad of North Carolina. After a time, A sells his stock for \$150, and gains as much per cent. on his outlay as B invested. B also sells his stock and gets \$12½ more than he gave for it, but his gain per cent. on his outlay is only half as great as that of A. Required the amount invested by each. *Ans.* A, \$100; B, \$50.

Why is the negative solution rejected?

Verification. 50 per cent. on \$100 is \$50. And since A sold his stock for \$50 more than he gave for it, he gained 50 per cent. on the \$100 of outlay. B gained 25 per cent on his outlay, and 25 per cent. upon \$50, is \$12½.

41. Two travellers set out at the same time, the one from A, and the other from C, and travel towards

each other at uniform rates. After meeting at B, the traveller from A is a days in reaching C, and the traveller from C, c days in reaching A. How long was it after the time of starting until they met at B, and how long was each traveller in performing the distance A C.

Time of meeting, $\pm\sqrt{ac}$. Traveller from A, $a \pm \sqrt{ac}$; traveller from C, $c \pm \sqrt{ac}$.

What is the meaning of the negative sign in the values of the time? When only can the time occupied by the traveller from A be negative? How then will both expressions for the time occupied by the other traveller be affected, with the positive or negative sign?

42. Same problem as the last, except that the traveller from A is 9 days between the points B and C, and the traveller from C, 25 days between the points B and A.

Ans. Time of meeting, $\pm$ 15 days; time occupied by traveller from A, 24, or -6 days; time occupied by the other traveller, 40, or -10 days.

43. In a certain bank there are \$438 worth of 5 and 3 cent pieces, the number of the latter is exactly the square of the number of the former. Required the number of pieces of each kind.

Ans. 120 five cent pieces; 14,400 three cent pieces.

44. Required to find two numbers, such that their sum, their product, and the difference of their squares, may all be equal to each other.

Ans. $x' = \frac{3}{2} + \sqrt{\frac{5}{4}}$, $x'' = \frac{3}{2} - \sqrt{\frac{5}{4}}$; $y' = \frac{1}{2} + \sqrt{\frac{5}{4}}$, $y'' = \frac{1}{2} - \sqrt{\frac{5}{4}}$.

45. In the year 1706 the French made a descent upon Charleston; but "South Carolina," says Bancroft, "gloriously defended her territory, and, with very little loss, repelled the invaders." A certain number of the French were killed and wounded, and 100 were taken prisoners. The number of killed and wounded was to the number of uninjured, including the prisoners, as 1 to 3. And the square of the number that escaped in safety from the expedition, was to the square of the number killed and wounded, as $6\frac{1}{4}$ to 1. Required the number of invaders, and the number of killed and wounded.

Ans. 800 invaders, and 200 killed and wounded.

Verification. If 200 were killed and wounded, then 600 were uninjured, and 200 : 600 : : 1 : 3. And, since 100 were taken prisoners, 500 escaped without harm from the expedition, and $(500)^2 : (200)^2 : : 6\frac{1}{4} : 1$.

Why is the value connected with the negative sign of the radical rejected?

46. In the year 1842 South Carolina converted the citadel at Charleston, and the magazine at Columbia, into military academies, which were to be supported by the sum of money appropriated annually

previous to this time to a guard of soldiers. The interest upon this sum for 21 months, amounted to \$1960, and the square of the interest for 6 months exceeded the square of 100^{th} part of the sum by \$288,000. Required the sum appropriated annually to the military academies, and the rate of interest. *Ans.* \$16,000, sum; 7 per cent. interest.

47. A man in Cincinnati purchased 10,000 pounds of bad pork, at 1 cent per pound, and paid so much per pound to put it through a chemical process, by which it would appear sound, and then sold it at an advanced price, clearing \$450 by the fraud. The price at which he sold the pork per pound, multiplied by the cost per pound of the chemical process, was 3 cents. Required the price at which he sold it, and the cost of the chemical process.

Ans. He sold it at 6 cents per pound, and the cost of the process was $\frac{1}{2}$ cent per pound.

48. The fore wheel of a wagon makes 12 more revolutions than the hind wheel, in going 240 yards; but if the circumference of each wheel be increased one yard, the fore wheel will make only 8 more revolutions than the hind wheel, in the same space. Required the circumference of each.

Ans. Circumference of fore wheel, 4 yards; circumference of hind wheel, 5 yards.

49. In the year 1853 there were a certain number of Woman's Rights conventions held in the State of New York. If 6 be added to the number and the square root of the sum be taken, the result will be exactly equal to the number. Required the number. *Ans.* 3.

How does the negative solution arise? Why is it neglected?

50. A planter purchased a number of slaves for \$36,000. If he had purchased 20 more for the same sum, the average cost would have been \$150 less. Required the number of slaves, and their average price.

Ans. 60 slaves; average price, \$600.

51. A planter purchased a number of slaves for m dollars. If he had received n more for the same sum, their average price would have been c dollars less. Required the number of slaves and their average price.

Ans. $x' = -\frac{n}{2} + \sqrt{\frac{4nm + cn^2}{4c}}$, $x'' = -\frac{n}{2} - \sqrt{\frac{4nm + cn^2}{4c}}$;

x' and x'' expressing the number of slaves. Then, $\frac{m}{x'}$ and $\frac{m}{x''}$ will express the average price.

Now, when n is zero, the two values of x ought both to be infinite, indicating an absurdity. But the expressions will not point out an absurdity unless reduced to their lowest terms by extracting the indicated roots. Then the two values of x may be written $-\frac{n}{2} \pm \left(\frac{n}{2} + \frac{m}{c} - \frac{m^2}{c^2n}\right.$, plus any other terms containing the higher powers of x in the denominators.$\left.\right)$

Now, make $n = 0$, and the two values of x both become infinite. An expression can never be correctly interpreted unless it is reduced to its lowest terms, for, as in the present instance, there may be a common factor in all its terms, and a particular hypothesis made upon that common factor may lead to absurd results.

What do the values become when $e = 0$? what when $n = 0$ and $c = 0$?

52. The field of battle at Buena Vista is $6\frac{1}{2}$ miles from Saltillo. Two Indiana volunteers ran away from the field of battle at the same time; one ran half a mile per hour faster than the other, and reached Saltillo 5 minutes and $54\frac{6}{11}$ seconds sooner than the other. Required their respective rates of travel. *Ans.* 6, and $5\frac{1}{2}$ miles per hour.

53. The New York shilling is $12\frac{1}{2}$ cents. A merchant bought a quantity of cloth for \$60. The number of shillings which he paid per yard was to the number of yards he bought as 1 to $4\frac{8}{10}$. Required the number of yards and the price per yard.

Ans. 48 yards; 10 shillings per yard.

54. A grocer sold 1000 pounds of coffee and 1500 pounds of rice for \$240; but he sold 500 pounds more of rice for \$80 than he sold of coffee for \$60. Required the price per pound of the coffee and rice.

Ans. Coffee, 12 cents per pound; rice, 8 cents per pound.

55. A, B, and C, entered into partnership, and gained as much as their joint fund, wanting \$200. A's gain was \$240. He invested

$100 more than B; and the joint investment of B and C was $700. Required the gain per cent., and the amount invested by each partner.

Ans. 80 per cent. A's capital, $300; B's, $200; C's $500.

56. There is a vessel containing 25 gallons of wine; a certain quantity is drawn out, and its place supplied by water. As much is drawn out of the adulterated liquor as was first drawn out of the pure wine, and there is now but 16 gallons of pure wine left in the vessel. Required the quantity of pure wine drawn out each time.

Ans. First draught, 5 gallons; second draught, 4 gallons.

57. Same problem as last, except that the vessel contains m gallons of wine, and that, after four draughts, $\frac{1}{m}$ gallons of pure wine are left.

Ans. First draught, $m-\sqrt{m}$; second, $\frac{m-\sqrt{m}}{\sqrt{m}}$; third, $\frac{m-\sqrt{m}}{m}$; fourth, $\frac{m-\sqrt{m}}{\sqrt{m^3}}$.

These results can readily be verified; for, when the four draughts are taken from m, there will remain

$$\sqrt{m}+\frac{\sqrt{m}-m}{\sqrt{m}}+\frac{\sqrt{m}-m}{m}+\frac{\sqrt{m}-m}{\sqrt{m^3}}, \text{ which is equal to } \sqrt{m}$$

$$+1-\sqrt{m}+\frac{1}{\sqrt{m}}-1+\frac{1-\sqrt{m}}{\sqrt{m^2}}, \text{ or } \frac{1}{\sqrt{m}}+\frac{1-\sqrt{m}}{m}, \text{ or }$$

$$\frac{\sqrt{m}+1-\sqrt{m}}{m}, \text{ or } \frac{1}{m}.$$

The equation to be solved was really one of the fourth degree; but, owing to its peculiar form, it was readily reduced to an equation of the second degree. Only one of the four values of the unknown quantity have been used.

58. Same problem as last, except that the vessel contains 64 gallons of wine, and that, after four draughts, the $\frac{1}{64}$th of a gallon is left.

Ans. First draught, 56 gallons; second, 7; third, $\frac{56}{64}$; fourth, $\frac{56}{512}$.

Verify these results by adding them together, and subtracting their sum from 64.

59. Two cotton merchants rent a house for a certain sum, with the understanding that each shall pay in proportion to the number of bales of cotton he puts in the house. A puts in 500 bales, and B as many bales as makes his proportion of pay amount to \$200. B afterwards puts in 500 more bales, and then his proportion amounts to \$225. Required the sum paid for house rent, and the number of bales first put in by B. *Ans.* House rent, \$300. B first put in 1000 bales.

60. A gentleman has two sums of money at interest, amounting to \$1150. The larger sum, being put out $\frac{5}{6}$th per cent. less advantageously than the smaller, brings only the same amount of yearly interest. At the end of ten years, the smaller sum, added to its simple interest for the whole period, is to the larger sum, added also to its simple interest, as 11 is to $11\frac{1}{2}$. Required the two sums, and the per cent. on each.

Ans. \$550 at 10 per cent., and \$600 at $9\frac{1}{6}$ per cent.

61. A gentleman bought a rectangular piece of land, giving \$10 for every yard in its perimeter. If the same quantity of ground had been in a square shape, it would have cost him \$20 less. And, if he had bought a square piece of land, of the same perimeter as the rectangle, it would have contained $6\frac{1}{4}$ square yards more. Required the sides of the rectangle. *Ans.* 4 and 9 yards.

62. A and B, together, invested \$800 in a speculation. A's money was employed 3 months, and B's 5 months. When they came to settle, A's capital and profits amounted to \$451; B's amounted to \$375. Required the capital of each, and their gain per cent.

Ans. A's, \$440; B's, \$360. 10 per cent. gain.

63. The seventh page of a treatise on Analytical Geometry has 11 more lines than the twentieth page of a treatise on Optics; but the seventh page of the Analytical Geometry has 4 letters less in each line than there are lines in the twentieth page of the Optics, whilst the twentieth page of the Optics has as many letters in each line as there are lines in the seventh page of the Optics. The total number of letters on both pages is 3542. Required the number of lines and letters in each page.

Ans. Analytical Geometry, 46 lines, and 42 letters in each line; Optics, 35 lines, and 46 letters in each line.

64. Required to divide the quantity, a, into two such parts, that the sum of the quotients, arising from dividing each part by the other, shall be equal to m.

Ans. First part, $\frac{a}{2} + \frac{a}{2}\sqrt{\frac{m-2}{m+2}}$, or $\frac{a}{2} - \frac{a}{2}\sqrt{\frac{m-2}{m+2}}$; second part, $\frac{a}{2} - \frac{a}{2}\sqrt{\frac{m-2}{m+2}}$, or $\frac{a}{2} + \frac{a}{2}\sqrt{\frac{m-2}{m+2}}$.

Are there four independent values? When only will these values be real? When will one always be negative? When will the two parts be equal?

65. Required to divide 10 into two such parts, that the sum of the quotients, arising from dividing each part by the other, shall be equal to $2\frac{16}{21}$. *Ans.* 7 and 3, or 3 and 7.

66. Required to divide the number 100 into two such parts, that the sum of the quotients, arising from dividing each part by the other, shall be equal to 2. *Ans.* 50 and 50.

67. Required to divide 15 into two such parts, that the sum of the quotients, arising from dividing each part by the other, shall be equal to $9\frac{1}{9}$. *Ans.* $13\frac{1}{2}$, and $1\frac{1}{2}$.

68. Four numbers are in a continued proportion, each number being an exact number of times greater than that which precedes. The difference between the means is 8, and the difference between the extremes 28. Required the sum of the means, and the terms of the proportion.

Ans. Sum of the means, 24. The numbers are 4, 8, 16 and 32.

In this example, let the unknown quantity be the sum of the means.

69. Same example as preceding, except making the difference of the means, a, and the difference of the extremes, b.

Ans. Sum of the means $+ a\sqrt{\frac{b+a}{b-3a}}$; greater mean $\frac{a + a\sqrt{\frac{b+a}{b-3a}}}{2}$; smaller, $\frac{a\sqrt{\frac{b+a}{b-3a}} - a}{2}$; greater extreme $\frac{b}{2} + \frac{1}{2}\sqrt{\frac{4a^3 + b^2(b-3a)}{b-3a}}$; smaller $-\frac{b}{2} + \frac{1}{2}\sqrt{\frac{4a^3 + b^2(b-3a)}{b-3a}}$.

The negative values have been rejected.

When will these values become infinite? When imaginary?

70. There are two numbers, such that the square of the first added to their product is equal to m, and the square of the second added to their product is equal to n. What are the numbers?

Ans. First, $\pm \frac{m}{\sqrt{m+n}}$; second, $\pm \frac{n}{\sqrt{m+n}}$.

71. Two numbers are to each other as m to n, and the square of the first added to the product of its first power by m, is equal to the square of the second, added to the product arising from multiplying its first power by n. What are the numbers?

Ans. First, 0, or $-m$; second, 0, or $-n$.

72. The sum of the squares of two numbers diminished by twice their product and by twice the first number, is equal to unity; and the sum of their squares added to the first power of the second number, is equal to twice the product of the numbers. What are the numbers?

Ans. First, 0, or $-\frac{4}{9}$; second, -1, or $-\frac{1}{9}$.

TRINOMIAL EQUATIONS.

353. A trinomial equation is one of the form, $x^{2n} + x^n = q$, involving but one unknown quantity and three terms, and having the unknown quantity in one term affected with an exponent double of that with which it is affected in the other. A trinomial equation then contains two terms, in which the unknown quantity enters, and an absolute term.

Let it be required to solve the equation, $x^4 + x^2 = 20$.

Let $x^2 = y$. Then the equation becomes $y^2 + y = 20$.

The last equation gives $y = +4$ and $y = -5$. But, $x^2 = y$. Hence, $x^2 = 4$, or -5. Then, $x' = +2$, $x'' = -2$, $x''' = +\sqrt{5}$, $x'''' = -\sqrt{-5}$. Either of the four values will satisfy the given equation, $x^4 + x^2 = 20$. The substitution of the last value gives $(-\sqrt{-5})^4 + (-\sqrt{-5})^2 = 20$, or, $25 - 5 = 20$, a true equation.

Solve the equation, $x^6 - x^3 = 702$. Make $x^3 = y$. Then the equation becomes $y^2 - y = 702$. From which, $y = +27$, or, -26. But $x^3 = y$. Hence, $x = \sqrt[3]{27} = 3$, and $x = \sqrt[3]{-26}$.

There are really four more values for x, since, as will be shown hereafter, the number of values is exactly equal to the degree of the equation. But the method of determining the other values belongs properly to the general theory of equations. The two values found can be readily verified.

The examples given are of a simple character. The equations were already of the proposed form. But it frequently happens that an artifice must be employed to put the equation under the form of $x^{2n} + x^n = q$. Take, as an example, $x^2 + \sqrt{x^2 + 9} = 21$. Add 9 to both members, and we have $x^2 + 9 + \sqrt{x^2 + 9} = 30$. Make $x^2 + 9 = y^2$, then, $\sqrt{x^2 + 9} = y$, and the equation becomes $y^2 + y = 30$. From which we get, $y = +5$, or -6. Then, $x^2 + 9 = 25$, or, $x^2 + 9 = 36$. The values of x are $+4$, -4, $+\sqrt{27}$, and $-\sqrt{27}$. The negative sign of the radical, $\sqrt{x^2 + 9}$, must be taken in connection with the last two values. This is indicated by the equation, $\sqrt{x^2 + 9} = y = -6$.

Again, take $\dfrac{(x^2 + 9)^2}{x} + 1 = \dfrac{x^2 + 9}{x} + 91$.

This may be written, $\dfrac{(x^2 + 9)^2}{x} - \dfrac{(x^2 + 9)^1}{x} = 90$.

Let $\dfrac{x^2 + 9}{x} = y$. The equation then becomes $y^2 - y = 90$. Hence, $y = +10$, or -9. Then, $\dfrac{x^2 + 9}{x} = 10$, and $\dfrac{x^2 + 9}{x} = -9$. These equations gives the four values, $+9$, $+1$, and $-\dfrac{9}{2} + \frac{1}{2}\sqrt{45}$, $-\dfrac{9}{2} - \frac{1}{2}\sqrt{45}$.

The equation, $\sqrt[4]{x + 1} + \sqrt[4]{x^2 + 2x + 1} = 20$, may be changed into $\sqrt{x + 1} + \sqrt[4]{x + 1} = 20$. Let $\sqrt[4]{x + 1} = y$. Then $\sqrt{x + 1} = y^2$, and the equation becomes $y^2 + y = 20$. From which, $y = 4$, or, -5. Then $\sqrt{x + 1} = 16$, and $\sqrt{x + 1} = 25$. Hence, $x = 255$, or, $x = 624$.

The equation, $x^2 + \sqrt{x^2 + 56} = 34$, may be placed under the proposed form by adding 56 to both members. Then, $x^2 + 56 + \sqrt{x^2 + 56} = 90$. Let $\sqrt{x^2 + 56} = y$. The equation then becomes $y^2 + y = 90$. From which, $y = +9$, or, -10. Then $x = +5$, -5, $+\sqrt{44}$, $-\sqrt{44}$.

These illustrations are sufficient to explain the spirit of the process. No general rule can be given. Any modification may be made upon the equation that will place it under the proposed form.

GENERAL EXAMPLES.

1. Solve the equation, $x^{\frac{2}{3}} + x^{\frac{1}{3}} = 12$.

Ans. $x = +27$, or -64.

2. Solve the equation, $x + x^{\frac{1}{2}} = 6$. *Ans.* $x = 4$, or 9.

3. Solve the three equations,

$$z^2 + zy + y^2 = 1900 \text{ (A).}$$
$$x^2 + xz + z^2 = 1300 \text{ (B).}$$
$$y^2 + xy + x^2 = 700 \text{ (C).}$$

Subtracting (B) from (A), and (C) from (B), and factoring, we get, $y + z + x = \dfrac{600}{y - x}$ (D), and $y + z + x = \dfrac{600}{z - y}$ (E).

By equating (D) and (E), we get $y = \dfrac{x + z}{2}$. This value of y, substituted in (D), gives $z + x = \dfrac{800}{z - x}$, or, $z^2 - x^2 = 800$; from which $z^2 = x^2 + 800$. Substituting for z its value in B, we get $x^2 + x\sqrt{x^2 + 800} + x^2 + 800 = 1300$. From which, $x\sqrt{x^2 + 800} = 500 - 2x^2$. Squaring and reducing, there results, $3x^4 - 2800x^2 = -250000$, or, $x^4 - \dfrac{2800x^2}{3} = -\dfrac{250000}{3}$.

The combination has led to a trinomial equation, which, when solved, will give, for one system of values, $x = 10$, $y = 20$, and $z = 30$.

4. Solve the equations,

$$y^2 + xy + x^2 = 7,$$
$$y^2 - xy - x^2 = -5.$$

Ans. $x = +2, -3$, $y = +1 -1$.

5. Solve the equations,

$$x + y + z = 6,$$
$$x^2 + y^2 + z^2 = 14,$$
$$xz = 3.$$

Ans. $y = 2$, $x = 1$, $z = 3$.

One system of values only given.

6. Solve the equations,

$$y^3 + x^2 = 2600,$$
$$y^2 - 2y = 2x.$$

Ans. $y = 10$, and $x = 40$.

One set of values only given.

7. Solve the equations, $y^5 + x^2 = y^3 + y^4 + 8$,

$$y^3 - 2y^2 = 2x.$$

Ans. One set of values, $y = 2$, and $x = 6$.

8. Solve the equation, $\sqrt{x^2 + 4} + \sqrt[4]{x^2 + 4} = 6$.

Ans. One set of values, $x' = + \sqrt{12}$, $x'' = - \sqrt{12}$.

9. Solve the equation, $\sqrt{y^2 - y} + \sqrt[4]{y^2 - y} = 3\sqrt{10} + \sqrt[4]{90}$.

Ans. One set of values, $y = 10$, or $y = -9$.

The positive value of the radical only is taken.

10. Solve the equation, $x^2 + 7 + \sqrt{x^2 + 7} = 20$.

Ans. $x' = +3$, $x'' = -3$, $x''' = + \sqrt{18}$, $x^{iv} = - \sqrt{18}$.

PROBLEM OF THE LIGHTS.

354. Two lights are placed on the same indefinite right line, the one shining with an intensity represented by a, the other with an intensity represented by b; the problem is to find, on this indefinite line, the point or points of equal illumination — assuming a principle of optics, that the intensity of a light varies inversely with the square of the distance from that light.

Let A be the position of the first light; B, that of the second.
Let m = AB, the distance between the lights.
Let a = intensity of first light, at one foot from A.
Let b = intensity of second light at one foot from B.
Let I be the unknown point of equal illumination.
Let B I = $m - x$ = distance of same point from B.

Now, in accordance with the assumed optical principle, the intensity of the first light, two feet from A, will be expressed by $\frac{a}{4}$ (for $2^2 : 1^2 :: a : \frac{a}{4}$), at three feet, by $\frac{a}{9}$, and at x feet, by $\frac{a}{x^2}$, and this last expression will represent the intensity of the first light at the unknown point, I. In like manner, the intensity of the second light, at B, will be denoted by $\frac{b}{(m - x)^2}$. But, by the conditions of the problem, the intensities of the two lights must be equal at I. Hence, we have $\frac{a}{x^2} = \frac{b}{(m - x)^2}$, which may be changed into $\frac{x^2}{(m - x)^2} = \frac{a}{b}$, and, by extracting the root, we get $\frac{x}{m - x} = \pm \frac{\sqrt{a}}{\sqrt{b}}$. We will, for convenience, call x' that value of x which is connected with the positive sign in the second member, and we will call x'' that which is connected with the negative sign. Hence, $\frac{x'}{m - x'} = \frac{\sqrt{a}}{\sqrt{b}}$, and $\frac{x''}{m - x''} = - \frac{\sqrt{a}}{\sqrt{b}}$. And solving these equations, we get $x' = \frac{m\sqrt{a}}{\sqrt{a} + \sqrt{b}}$, and $x'' = \frac{m\sqrt{a}}{\sqrt{a} - \sqrt{b}}$.

Now, since there are two distinct values for x, we conclude that in general there will be two points of equal illumination. By this we do not mean that there will be two points of equal brilliancy, but that there will be two points where the intensities of the two lights will be equal to each other. By subtracting x' and x'' in succession from m, we get

$$m - x' = \frac{mb}{\sqrt{a} + \sqrt{b}}, \text{ and } m - x'' = - \frac{m\sqrt{b}}{\sqrt{a} - \sqrt{b}}.$$

Hence, we have the system of values:

$x' = \frac{m\sqrt{a}}{\sqrt{a} + \sqrt{b}}$, distance from A to first point of equal illumination.

$m - x' = \frac{m\sqrt{b}}{\sqrt{a} + \sqrt{b}}$, distance from B to same point.

$x'' = \frac{m\sqrt{a}}{\sqrt{a} - \sqrt{a}}$, distance from A to second point of equal illumination, if there be one.

$m - x'' = - \frac{m\sqrt{a}}{\sqrt{a} - \sqrt{b}}$, distance from B to same point.

Now, these expressions have been deduced upon the supposition that the point, I, was between A and B, and, consequently, we have assumed that the distance A I is positive, when estimated on the right of A, and the distance B I positive, when estimated on the left of B. Hence, if either x' or x'' becomes negative in consequence of any imposed condition, the corresponding point of equal illumination will be found on the left of A. And, in like manner, if either $m - x'$ or $m - x''$ becomes negative, the point will be found on the right of B.

We will begin the discussion by supposing $a = b$. Then, $x' = \frac{m\sqrt{a}}{\sqrt{a} + \sqrt{b}} = \frac{m\sqrt{a}}{2\sqrt{a}} = \frac{m}{2}$, or $\text{A I} = \frac{\text{A B}}{2}$. Hence, the point is half way between the lights, as it obviously ought to be. Now, $m - x'$, which expresses the distance from B to I, ought to give the same point. It does so, for $m - x' = \frac{m\sqrt{b}}{\sqrt{a} + \sqrt{b}} = \frac{m\sqrt{b}}{2\sqrt{b}} = \frac{m}{2}$. We will next examine whether there can be a second equally illuminated point, when the intensities of the two lights are the same. We have $x'' = \frac{m\sqrt{a}}{0}$ $= \infty$, and $m - x'' = -\frac{m\sqrt{b}}{0} = -\infty$. These values indicate that the second point is at an infinite distance on the right of B; or, in other words, that there is no such point at all. This ought to be so, for, when the two lights are of equal intensity, the one cannot throw its beams with the same power to a greater distance than the other, and this must be the case if there be a second point. The second values (x'' and $m - x''$) then denote impossibility. By going back to the equation of the problem, we see that it will become, when $a = b$, $x^2 = (m - x)^2$, an equation which can only be true when $m = 0$, or $x = \frac{m}{2}$. But $m = 0$ is contrary to the hypothesis; hence, $x = \frac{m}{2}$ is the only true value. The assumption then, that there was a second point (a being equal to b), was absurd, and led to ∞, the appropriate symbol of absurdity.

We will next suppose $a > b$.

Then, $x' = \frac{m\sqrt{a}}{\sqrt{a} + \sqrt{b}}$ becomes $> \frac{m}{2}$, because, if the denominator were $2\sqrt{a}$, the value of the fraction would be $\frac{m}{2}$, and since the denominator is less than $2\sqrt{a}$, the value is greater than $\frac{m}{2}$.

Hence, the point, I, is nearer B than A. In this case,

$m - x' < \frac{m}{2}$, for, were the denominator of the fraction $\frac{m\sqrt{b}}{\sqrt{a} + \sqrt{b}}$, exactly $2\sqrt{b}$, the value of the fraction would be $\frac{m}{2}$; but since the denominator is greater than $2\sqrt{b}$, the value is less than $\frac{m}{2}$. This result corresponds to the former, and places I nearer B than A; that is, nearer the light of feebler intensity, as it ought to do. We see that x'' is greater than m, for were the denominator of x'', the $\sqrt{a}$, its value would be equal to m, but as $\sqrt{a} - \sqrt{b}$ is less than $\sqrt{a}$, x'' is greater than m. Hence, the second point of equal illumination is beyond B. In this case, $m - x'' = -\frac{m\sqrt{b}}{\sqrt{a} - \sqrt{b}}$ becomes negative, since the numerator is negative and denominator positive, and, therefore, the second point must be on the right of B. The results then agree with each other, and agree with the fact. For, after passing the point I, the intensity of the 1st light becomes feebler than that of the 2d. At the point B, there is the greatest difference between their intensities. Beyond B, both lights diminish in brilliancy, but the 2d more rapidly than the 1st, because of its greater feebleness; and we at length reach a point, I′, where the illumination is equal.

Next, suppose $b > a$.

Then, $x' = \frac{m\sqrt{a}}{\sqrt{a} + \sqrt{b}} < \frac{m}{2}$. For, were the denominator $2\sqrt{a}$, the value of the fraction would be $\frac{m}{2}$, but as the denominator is $> 2\sqrt{a}$, the value of the fraction would be $< \frac{m}{2}$

Hence, the point, I, will be found nearer A than B. We have $m - x' = \frac{m\sqrt{b}}{\sqrt{a} + \sqrt{b}} > \frac{m}{2}$; for, were the denominator $2\sqrt{b}$, the value would be $= \frac{m}{2}$, but as the denominator is $< 2\sqrt{b}$, the value is $> \frac{m}{2}$. The results then agree, and place the point, I, nearer the feebler light. Again, we have $x'' = \frac{m\sqrt{a}}{\sqrt{a} - \sqrt{b}}$ affected with the sign minus, the numerator being positive, and denominator negative. The second point, I″, must then be on the left of A. By multiplying the numerator and denominator, of the value of $m - x''$, by minus unity, we will have $m - x'' = \frac{m\sqrt{b}}{\sqrt{b} - \sqrt{a}} > m$; for, were the denominator $= \sqrt{b}$, the value would be equal to m, and since the denominator is $< \sqrt{b}$, the value is greater than m. Hence, the second point is beyond A at I″, as has just been shown. Were the values of m, a, and b, given, the position of the point of equal illumination could be readily determined. Let $m = 20$ feet, $a = 16$, $b = 36$. Then $x' = 8$ feet, and $m - x' = 12$ feet, and the intensity of the two lights will be equal at the point corresponding to these distances. For, the intensity of the 1st will be $\frac{16}{(8)^2} = \frac{1}{4}$, and that of the 2d $\frac{36}{(12)^2} = \frac{1}{4}$

It can readily be shown that, between I and A, the intensity of the 1st light is greater than that of the 2d. Thus, at 2 feet from A, that of the 1st light will be expressed by 4, and that of the 2d by $\frac{1}{9}$. At the point I″, 40 feet on the left of A, and 60 feet from B, the intensities of both lights will be expressed by $\frac{1}{100}$th, and are, therefore, equal. And, by assuming other points, and calculating the corresponding intensities, we would find that they were only equal at I and I″.

Now, let $m = 0$, and a unequal to b, then x', $m - x'$, x'', and $m - x'' = 0$. These are plainly absurd solutions, for the two lights, shining with unequal brilliancy, cannot, equally, illumine the point at which both are placed. By recurring to the equation of the problem, it becomes, when $m = 0$, $\frac{x^2}{x^2} = \frac{a}{b}$. This equation can only be true when $a = b$, which is contrary to hypothesis.

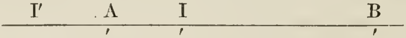
Now, let $m = 0$, and $a = b$.

Then, $x' = 0$, $m - x' = 0$; $x'' = \frac{0}{0}$, and $m - x'' = \frac{0}{0}$. The first two values (x' and $m - x'$,) show that the point at which the lights are placed, is equally illuminated. The second two (x'' and $m - x''$,) are the symbols of indetermination. This ought to be so, for when the equal lights are placed together, the points, at one foot, two feet, three feet, &c., are as much irradiated by one light as the other. There is, therefore, no particular determinate point at which the intensities are equal, and we say that the problem is indeterminate. By recurring to the equation of the problem, $\frac{x^2}{(m-x)^2} = \frac{a}{b}$, we see that when $m = 0$, and $a = b$, it reduces to an identical equal equation, $x^2 = x^2$, which can be satisfied for any values of x.

Whenever we get $\frac{0}{0}$ as a solution to a problem, we can tell, by recurring to the original equation, whether it indicates indetermination. If the particular hypothesis, which reduces the solution to $\frac{0}{0}$, also reduces the original equation to an identical form, we have a case of indetermination. But, if the given equation do not reduce to an identity, we have a vanishing fraction.

Another test, also, may be employed. If the value becomes $\frac{0}{0}$ in consequence of a single hypothesis, we know, certainly, that we have a vanishing fraction, and not an indeterminate solution. The preceding values become $\frac{0}{0}$ in consequence of the two hypotheses, $m = 0$, and $a = b$, and might or might not be vanishing fractions. Thus, the expression, $\frac{(x^2 - a^2)(x - b)}{(x - a)(x^2 - b^2)}$, is a double vanishing fraction for $x = a$, and $x = b$.

Now, suppose the first light extinguished; then, $a = 0$; and we find x' and $x'' = 0$, and $m - x'$, and $m - x'' = m$. These solutions, at first, seem absurd, since they indicate that the point of equal illumi-



nation is at A. But, suppose the light at A to be very feeble, there will then be two points of equal illumination, I and I′, very near to and on opposite sides of A. Make the first light still more feeble, and the two points, I and I′, will approach nearer to A; and, finally, when that light is extinguished, they will unite at A.

Next, suppose both lights extinguished, or $a = 0$, and $b = 0$. Then, x', x'', $m - x'$, and $m - x''$ become $\frac{0}{0}$, indeterminate, as they ought to be.

Suppose $m = \infty$, or that the lights are infinitely distant from each other. Then, if a and b are finite, the values all become infinite, as they ought, since there can be no point of equal illumination. The equation of the problem, $\frac{a}{x^2} = \frac{b}{(m-x)^2}$, can be satisfied, when $m = \infty$, by making $x = 0$, which places the point of equal illumination at A; or, by making $x = m$, which places the point of equal illumination at B.

Suppose, $a = \infty$; then, x' and x'' both become equal to m, since $\sqrt{b}$ may be neglected in the denominators of the fractions, and the point of equal illumination is then at B. The values of $m - x'$ and $m - x''$ become zero, and indicate the same point. This ought to be so; for, by making the intensity of one light infinitely great, we make that of the other relatively infinitely small, and we then ought to get the same result as we did when one of the lights was supposed to be extinguished.

UNDETERMINED COEFFICIENTS.

355. The method of undetermined coefficients is used to develop algebraic fractions into a series, and to determine the value of the constants which enter into identical equations; that is, equations which can be satisfied by any value whatever, attributed to the unknown quantity.

For the development of fractions we assume the form of development; and we are governed in our assumption of this form by our knowledge of what it ought to be. Thus, if we proceed to expand $\frac{a}{a+x}$ by the ordinary process of division, we obtain $\frac{a}{a+x} = 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \frac{x^5}{a^5} +$, &c. The quotient is an infinite series, arranged according to the ascending powers of x, and having x in each term affected with a positive and an entire exponent. The first term may be supposed to contain x, affected with a zero exponent, and the coefficients of x are then $1 - \frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^3} +$ &c.; and it is evident that these coefficients are entirely independent of the particular values that may be attributed to x. They are still $1, -\frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^3} +$ &c.;

when $x = 0, 1, 2, 1000$, anything whatever. We may remark, in this connection, that the independence of the coefficients upon the value of the letter with which they are associated is not confined to the expansion of fractions, but is true in all developments whatever. Thus, $(a + x)^2 = a^2 + 2ax + x^2$; the coefficients are a^2, $2a$, and 1, and these coefficients will not be altered by any change in the value of x.

We have seen, by performing the division, that the exponents of x, in the development of $\frac{a}{a + x}$, must be positive and entire, and that the coefficients must be independent of x. If, then, we were required to assume the form of development of this fraction, or any like it, we would know that the assumed form must fulfil the required conditions. Now, since a may be regarded as the coefficient of x^0, the fraction, $\frac{a}{a + x}$, may be considered as arranged with reference to x, both in the numerator and denominator; and we have seen that a fraction so arranged, beginning with the zero power of x, gives positive and entire exponents for x in the quotient. This law can readily be shown to be general, and it is not even necessary that the zero power of x should enter into the numerator. If the exponents of x in the numerator are all positive and entire, and the terms arranged according to the ascending powers of x, and if the denominator is also arranged in like manner, and contains a *constant*, a', that is, a term involving x^0, then all the exponents will be necessarily positive and entire. For, take the general fraction, $\frac{a + bx^n + cx^m}{a' + b'x^p + c'x^q} = A + Bx^{\frac{m}{n}} + Cx^{-r} + \&c.$, (N), in which the second member represents the development of the fraction. Now, since a fraction and its development must constitute an identical equation, it is plain that, from the nature of such equations, (N) must be true when $x = 0$. But Cx^{-r}, which represents the term (if any) containing a negative exponent, can be written $\frac{C}{x^r}$, and is equal to infinity when $x = 0$. Hence, the whole second member is infinite. But, $x = 0$; reduce the first member to $\frac{a}{a'}$; n, m, p, and q being supposed positive. Now, if $a = 0$, we have $\frac{0}{a'} = 0 =$ second member $= \infty$, which is absurd. If a be not zero, still we have $\frac{a}{a'} = \infty$, or a finite quantity equal to an infinite, which is absurd.

If, then, all the exponents in the numerator and denominator are positive and entire, and the denominator contains a constant, or both numerator and denominator contain constants, all the exponents of the development must be positive. We will now show that all the exponents of x must also be entire. We have represented, by $Bx^{\frac{m}{n}}$, the term, if any, involving a fractional exponent. Now, suppose $\frac{m}{n} = \frac{1}{2}$. Then, $Bx^{\frac{m}{n}} = Bx^{\frac{1}{2}} = B\sqrt{x}$. But every square root has two distinct values. Hence, $B\sqrt{x}$ has two distinct values, and the whole second member has two distinct values. But no fraction containing only entire exponents can give two quotients. The assumption, then, of a fractional exponent in the development is absurd. If $\frac{n}{m} = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$, &c., the term $Bx^{\frac{m}{n}}$ would have three values and three quotients, which is absurd. So, if $\frac{n}{m} =$ any fraction, there would be more than one quotient.

We are now prepared to assume the form of development of any algebraic fraction, arranged according to the powers of a certain letter, and having that letter in the first term of the numerator affected with an exponent equal to, or greater than that of this letter in the first or last term of the denominator. The exponents of the arranged letter are also assumed to be positive and entire, both in the numerator and denominator.

Let us then place $\frac{a}{a + x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$, in which A, B, C, D, &c., are independent of x, and in which all the exponents of x are positive and entire. A is the representative of that term which does not contain x, or contains x affected with a zero exponent. There is generally such a term in a development, and there must, of course, be a representative of that term. Clearing of fractions and arranging according to the ascending powers of x, we have

$$a = Aa + \left.\begin{array}{r} aB \\ +A \end{array}\right| x + \left.\begin{array}{r} aC \\ +B \end{array}\right| x^2 + \left.\begin{array}{r} aD \\ +C \end{array}\right| x^3 + \left.\begin{array}{r} aE \\ +C \end{array}\right| x^4 + \&c. \quad (M).$$

Now, since the coefficients are supposed to be independent of x, their values will not be affected by making $x = 0$. If then we can determine these values when $x = 0$, they will be true when $x = 1, 2, 3$; anything whatever. Making $x = 0$, we have $a = Aa$. Hence, $A = 1$.

Now, since $a = \mathrm{A}a$, these two terms cancel each other, and (M) becomes

$$0 = \begin{array}{r|l|l|l|l} a\mathrm{B} & x + a\mathrm{C} & x^2 + a\mathrm{D} & x^3 + a\mathrm{E} & x^4 + \&c. \\ +\mathrm{A} & \quad +\mathrm{B} & \quad +\mathrm{C} & \quad +\mathrm{D} & \end{array}$$

Dividing both members of this equation by x, which we have a right to do, we get

$$0 = \begin{array}{r|l|l|l} a\mathrm{B} + a\mathrm{C} & x + a\mathrm{D} & x^2 + a\mathrm{E} & x^3 + \&c.\ (\mathrm{N}). \\ +\mathrm{A} \quad +\mathrm{B} & \quad +\mathrm{C} & \quad +\mathrm{D} & \end{array}$$

Making, again, $x = 0$, we have left, $0 = a\mathrm{B} + \mathrm{A}$. Hence, $\mathrm{B} = -\frac{\mathrm{A}}{a} = -\frac{1}{a}$. Now, since $a\mathrm{B} + \mathrm{A}$ has been found to be zero, they may be stricken out of (N), and that equation will become

$$0 = \begin{array}{r|l|l|l} a\mathrm{C} & x + a\mathrm{D} & x^2 + a\mathrm{E} & x^3 + \&c. \\ +\mathrm{B} & \quad +\mathrm{C} & \quad +\mathrm{D} & \end{array}$$

And again, dividing out by x, we have

$$0 = \begin{array}{r|l|l} a\mathrm{C} + a\mathrm{D} & x^2 + a\mathrm{E} & x^2 + \&c.\ (\mathrm{P}). \\ +\mathrm{B} \quad +\mathrm{C} & \quad +\mathrm{D} & \end{array}$$

Making $x = 0$ in (P), we have left, $0 = a\,\mathrm{C} + \mathrm{B}$. Hence, $\mathrm{C} = -\frac{\mathrm{B}}{a} = +\frac{1}{a^2}$. Omitting the term, $a\mathrm{C} + \mathrm{B} = 0$, in equation (P), and dividing again by x, we have

$$0 = \begin{array}{r|l} a\mathrm{D} + a\mathrm{E} & x + \&c.\ (2). \\ +\mathrm{C} \quad +\mathrm{D} & \end{array}$$

Making $x = 0$, we have left, $0 = a\mathrm{D} + \mathrm{C}$. Hence, $\mathrm{D} = -\frac{\mathrm{C}}{a} = -\frac{1}{a^3}$. Dividing (2) by x, and again making $x = 0$, we have $a\mathrm{E} + \mathrm{D} = 0$. Hence, $\mathrm{E} = -\frac{\mathrm{D}}{a} = +\frac{1}{a^4}$. Now, if we substitute for A, B, C, &c., their found values, in the original equation, $\frac{a}{a+x} = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \mathrm{E}x^4 + \&c.$, we will have $\frac{a}{a+x} = 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \frac{x^5}{a^5} + \&c.$; the same result that we obtained by division By transposing a to the second member in equation (M), we have

$$0 = \begin{array}{r|l|l|l|l} \mathrm{A}a & x^0 + a\mathrm{B} & x + a\mathrm{C} & x^2 + a\mathrm{D} & x^3 + \&c. \\ -a & \quad +\mathrm{A} & \quad +\mathrm{B} & \quad +\mathrm{C} & \end{array}$$

And, since we have found $Aa - a = 0$, $aA + A = 0$, $aC + B = 0$, $aD + C = 0$, &c., we conclude, that, *if we have an equation whose first member is zero, and whose second member contains all its terms arranged according to the ascending powers of a certain letter, the exponents of this letter being all positive and entire, and its coefficients independent of it, these coefficients will be separately equal to zero.*

This is the first enunciation of the principle of undetermined coefficients, and ought to be remembered.

The second enunciation, (which is an immediate consequence of the first), is as follows: if we have an equation of the form, $a + bx + cx^2 + dx^3 + ex^4 +$ &c., $= a' + b'x + c'x^2 + d'x^3 + e'x^4 +$ &c., which is satisfied for any value of x, then the coefficients of the like powers of x in the two members will be respectively equal to each other.

For, by transposition, we have $0 = (a' - a)\,x^0 + (b' - b)\,x + (c' - c)\,x^2 + (d' - d)\,x^3 + (e' - e)\,x^4 +$ &c. And, since these coefficients are independent of x, we must have, by what has just been shown, $a' - a = 0$, $b' - b = 0$, $c' - c = 0$, $d' - d = 0$, $e' - e = 0$, &c. Hence, $a' = a$, $b = b$, $c' = c$, $d = d$, $e' = e$, as enunciated.

The preceding equation, $a + bx + cx^2 + dx^3 + ex^4 +$ &c. $= a' + b'x + c'x^2 +$ &c., and all other equations to which the second enunciation is appliable, belong to a class of equations called identical equations, the two members of which differ only in form.

It is, in general, most convenient to develop expressions in accordance with the second enunciation.

Required the development of $\frac{1 - x}{1 + x}$.

Let $\frac{1 - x}{1 + x} = A + Bx + Cx^2 + Dx^3 + Ex^4 +$ &c.

Clearing of fractions, we have $1 - x = A + (B + A)\,x + (C + B)\,x^2 + (C + D)\,x^3 + (E + D)\,x^4 +$ &c. By placing $x = 0$, we have $A = 1$, and we might proceed as we did with the fraction, $\frac{a}{a + x}$, continually dividing by x, and making $x = 0$ in the resulting equation. But, by the second enunciation, we have at once $A = 1$, $B + A = -1$, $C + B = 0$, $C + D = 0$, and $E + D = 0$. From which, $A = 1$, $B = -2$, $C = +2$, $D = -2$, $E = -2$. And, substituting these values of A, B, C, D, E, &c. in the equation, $\frac{1 - x}{1 + x} = A + Bx + Cx^2 +$ &c., we get $\frac{1 - x}{1 + x} = 1 - 2x + 2x^2 - 2x^3 + 2x^4 -$ &c.

This is the same result that we would obtain by actually performing the indicated division.

We have assumed the form of development to be $A + Bx + Cx^2 +$ &c., in which the exponents are all positive and entire. Now, if we have an expression whose development must necessarily contain negative or fractional exponents, it would be absurd to place it equal to $A + B + x + Cx^2 +$ &c., and the result will make manifest the absurdity by the symbol, ∞. Suppose it be required to develop $\frac{1}{x - x^2}$. It is plain that the first term of the development is x^{-1}; if, then, we attempt to develop the expression by the method of undetermined coefficients, we commit an absurdity, and that absurdity ought to be made manifest in the result by the appropriate symbol, ∞.

Let, then, $\frac{1}{x - x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 +$ &c.

From this we get $1 = Ax + (B - A)x^2 + (C - B)x^3 + (D - C)x^4 +$ &c.

Now, in accordance with the first enunciation, make $x = 0$, and we get $1 = 0$, an absurd result. But, if we first divide both members by x, and then make $x = 0$, we will have $\frac{1}{0} = \infty = A$, and the absurdity is pointed out by its appropriate symbol.

The above expression can be developed by changing its form. Decompose $\frac{1}{x - x^2}$ into its factors, $\frac{1}{x} \times \frac{1}{1 - x}$, then develop $\frac{1}{1 - x}$ by the method of undetermined coefficients, and multiply the development by $\frac{1}{x}$. By developing, we find $\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 +$ &c.; consequently, $\frac{1}{x - x^2} = \frac{1}{x} \times \frac{1}{1 - x} = x^{-1} + x^0 + x + x^2 + x^3 + x^4 +$ &c. The series is evidently arranged according to the ascending powers of x. For most expressions, a slight inspection will show whether their development will contain negative or fractional exponents, and, consequently, whether the assumed form is right. Thus, $\frac{1}{1 - x^{-2}}$ cannot be placed equal to $A + Bx + Cx^2$, &c., because its development must contain negative exponents; $\frac{1}{1 + x^{\frac{1}{2}}}$ cannot be placed equal to the assumed form, because its development must contain fractional exponents These expressions, and others like them, can, however, be expanded into a series by changing their form. Place $x^{-2} = z$, then, $\frac{1}{1-x^{-2}} = \frac{1}{1-z}$.

Let $1 - z = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$

We will find $\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \&c$

Now, replace z by its value, and we have $\frac{1}{1-x^{-2}} = 1 + x^{-2} + x^{-4} + x^{-6} + x^{-8} + \&c.$

In like manner, place $x^{\frac{1}{2}} = z$. Then, $\frac{1}{1+x^{\frac{1}{2}}} = \frac{1}{1+z} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$ We will find by the method of undetermined coefficients, $\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4, \&c.$; and, replacing z by its value, we have $\frac{1}{1+x^{\frac{1}{2}}} = 1 - x^{\frac{1}{2}} + x - x^{\frac{3}{2}} + x^2 + \&c.$

FAILING CASES.

356. Any fraction, all of whose exponents are positive and entire in the letter according to which the series is to be arranged, can be developed, provided the denominator contains a constant. Again, any fraction, all of whose exponents are positive and entire in the letter according to which the series is to be arranged, can be developed; provided, that there is one term, at least, in the denominator, the exponent of whose arranged letter is equal to or less than the exponent of the same letter in, at least, one term of the numerator. A fraction that fulfils either of the foregoing conditions can always be developed, and the development will be the same as the quotient. The reason of this is plain; the method of undetermined coefficients gives a development for fractions, which is the same that would be obtained by actual division, beginning with the lowest power of the arranged letter. Now, a fraction that fulfils either of the foregoing conditions, will, when expanded by division, necessarily give positive and entire exponents in the arranged letter, when the division has been begun, with the numerator and denominator arranged according to the lowest power of this letter.

Division would give us two quotients for $\frac{x^2}{x+1}$, according as we made x, or 1, the first term of the divisor; but the method of undetermined coefficients gives but one development, which is the same as that obtained by arranging the numerator and denominator with reference to the lowest power of x. Therefore, the formula, which is applicable only to positive and entire exponents, must fail, when the least exponent of x in the numerator is less than the least exponent of x in the denominator. Thus, $\frac{x^2 + a^2}{x^2 + x}$ cannot be developed, because a^2, which involves the lowest power of x in the numerator, gives a quotient affected with a negative exponent when divided by x, the corresponding term of the denominator.

GENERAL EXAMPLES.

1. Develop $1 + x$. *Ans.* $1 + x$.

2. Develop $\frac{1}{1 + x + x^2}$ into a series.
Ans. $1 - x + x^3 - x^4 + x^6 - x^7 + x^8 - \&c.$

3. Develop $\frac{1 - 5x}{1 - 4x}$ into a series.
Ans. $1 - x - 4x^2 - 16x^3 - 64x^4 - 256x^5 - 1024x^6 - \&c.$

4. Develop $\frac{1 - x^2}{1 + x}$ into a series. *Ans.* $1 - x$.

5. Develop $\frac{x^{-2}}{1 - x^{-2}}$ into a series.
Ans. $x^{-2} + x^{-4} + x^{-6} + x^{-8} + x^{-10} + \&c.$

6. Develop $\frac{x^{\frac{1}{3}}}{1 - x^{\frac{1}{3}}}$ into a series.
Ans. $x^{\frac{1}{3}} + x^{\frac{2}{3}} + x + x^{\frac{4}{3}} + x^{\frac{5}{3}} + x^2 + \&c.$

7. Develop $\frac{1 + x}{1 - 2x}$ into a series.
Ans. $1 + 3x + 6x^2 + 12x^3 + 24x^4 + 48x^5 + \&c.$

8. Develop $\frac{1}{1 + x + x^2 + x^3}$ into a series.
Ans. $1 - x + x^4 - x^5 + x^8 - x^9 + x^{12} - x^{13} + \&c.$

9. Develop $\frac{1}{1 + x + x^2 + x^3 + x^4}$ into a series.

Ans. $1 - x + x^5 - x^6 + x^{10} - x^{11} + x^{15} - x^{16} + x^{20} - x^{21} + \&c.$

10. Develop $\frac{x^2}{a + b}$. *Ans.* $\frac{x^2}{a + b}$.

11. Develop $\frac{x^2}{x + 1}$ into a series.

Ans. $x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c.$

12. Develop $\frac{x^3 + x^2}{x + 5}$ into a series.

Ans. $\frac{x^2}{5} + \frac{4x^3}{25} - \frac{4x^4}{125} + \frac{4x^5}{625} - \&c.$

13. Develop $\frac{x^m - y^m}{x - y}$ into a series arranged according to the powers of y. *Ans.* $x^{m-1} + x^{m-2}y + x^{m-3}y^2 + x^{m-4}y^3 + \ldots . y^m$.

14. Develop $\frac{x^3 + x^2}{x^2 + x}$. *Ans.* x.

15. Develop $\frac{x^3 + 3x^2 + 3x + 1}{x + 1}$ into a series.

Ans. $x^2 + 2x + 1$.

16. Develop $\frac{x^3}{x^2 + x}$ into a series.

Ans. $x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c.$

17. Develop $\frac{x^4 + x^3 + x^2}{x^2 + x}$ into a series.

Ans. $x + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + \&c.$

18. Develop $\frac{x + x^2 + x^3 + x^5}{x^2 + x}$ into a series.

Ans. $1 + x^2 - x^3 + 2x^4 - 2x^5 + 2x^6 - 2x^7 + 2x^8 - 2x^9 + \&c.$

19. Develop $\frac{x^2 + x}{x + x^2 + x^3 + x^5}$ into a series.

Ans. $1 - x^2 + x^3 - x^4 + 2x^6 - 3x^7 + 2x \ + x^9 - 5x^{10}$.

20. Develop $\frac{x^3 + 3x^2 + 3x + 1}{x^2 + x}$ into a series. *Ans.* ∞.

The process fails. Why?

21. Develop $\frac{1 + 7x + x^2}{1 + x}$ into a series.

Ans. $1 + 6x - 5x^2 + 5x^3 - 5x^4 + 5x^5 - 5x^6 + \&c.$

22. Develop $\frac{x^2 - a^2}{x - a}$. *Ans.* $x + a$.

23. Develop $\frac{x^2 - a^2}{x + a^2}$ into a series.

Ans. $-1 + \frac{x}{a^2} + \frac{(a^2-1)x^2}{a^4} - \frac{(a^2 - 1)x^3}{a^6} + \frac{(a^2 - 1)x^4}{a^8} - \&c.$

24. Develop $\frac{x^{-4} + x^{-2} + 2}{x^{-2} + 1}$ into a series.

Ans. $2 - x^{-2} + 2x^{-4} - 2x^{-6} + 2x^{-8} - 2x^{-10} + \&c.$

Make $x^{-2} = z$. Develop, and replace z by x^{-2}.

Remarks.

357. It will be observed that, in most of the preceding examples, any term of the series after a certain number from the left, can be formed from the term immediately preceding, or from two or more preceding terms, according to some fixed law. Thus, in Example 2, every alternate term of the series is formed from that which precedes it, by multiplying by $-x$. The series in 5 and 6 are formed in the same way, the multipliers, or all the terms, being x^{-2} and $x\frac{1}{3}$. In 7, every term after the second is formed from that which precedes it, by multiplying by $2x$. In 8, we may regard the constant multiplier as $-x$, and omit every alternate set of two terms. In 9, the multiplier may be regarded as $-x$, and, when three terms occur together, these are omitted. In 11 and 16, the multiplier is $-x$. In 12, the multiplier, for all terms after the second, is $-\frac{1}{5}x$. In 13, the multiplier is $x^{-1}y$. In 17, the constant multiplier, after the second term, is $-x$. In 18, the multiplier, after the fourth term, is $-x$. In 19, there is no law of formation.

When the terms of a development are formed from those which precede according to some fixed law, the development is called a *recurring series.* Multiplication is not the only mode of forming the succeeding terms; sometimes they are formed by addition or subtraction, and even by a combination of two of these methods. Thus, $\frac{1}{1 - x - x^2 - x^3}$, expanded, becomes $1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + 24x^6 + \&c.$

The literal parts are formed by multiplying by x, the first two terms have the same coefficient, and each succeeding coefficient is equal to the sum of the three which precede it, &c. Again, $\frac{1 + 2x}{1 - x - x^2} = 1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + 29x^6 + 47x^7 + \&c.$, in which the coefficients after the second are equal to the sum of the two preceding coefficients.

PARTICULAR CASES.

358. A fraction involving irrational monomials may be treated as in the following example.

Required, the development of $\frac{x^{\frac{1}{2}} - x^{\frac{2}{3}}}{1 - x^{\frac{1}{2}}}$. Let $x = z^6$. Then, $x^{\frac{1}{2}} = z^3$, $x^{\frac{2}{3}} = z^4$, $z = x^{\frac{1}{6}}$. Then, the fraction becomes $\frac{z^3 - z^4}{1 - z^3}$, and, by the method of undetermined coefficients, $\frac{z^3 - z^4}{1 - z^3} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$ From which we get $A = 0$, $B = 0$, $C = 0$, $D = 1$, $E = -1$, $F = 0$, $G = 1$, $H = -1$, &c. Hence, $\frac{z^3 - z^4}{1 - z^3} = z^3 - z^4 + z^6 - z^7 + z^9 - z^{10} + \&c$, and, by replacing z by its value, we have $\frac{x^{\frac{1}{2}} - x^{\frac{2}{3}}}{1 - x^{\frac{1}{2}}} = x^{\frac{1}{2}} - {}^{\frac{2}{3}} + x - x^{\frac{7}{6}} + x^{\frac{3}{2}} - x^{\frac{5}{3}} + \&c.$

Since all examples of the same kind may be treated in the same way, we derive the

RULE.

Place the variable equal to a new variable, affected with an exponent equal to the least common multiple of the denominators of the exponents of the old variables. Develop the new fraction, and replace the new variable by its value in terms of the old.

EXAMPLES.

1. Expand $\frac{x^{\frac{1}{6}} - x^{\frac{2}{3}}}{1 - x}$ into a series.

Ans. $x^{\frac{1}{6}} - x^{\frac{2}{3}} + x^{\frac{7}{6}} - x^{\frac{5}{3}} + \&c.$

2. Expand $\frac{x^{\frac{1}{4}} - x^{\frac{1}{6}}}{1 + x}$. *Ans.* $x^{\frac{1}{4}} - x^{\frac{1}{6}} + x^{\frac{7}{6}} - x^{\frac{5}{4}} + \&c.$

359. Fractions involving parentheses, differing only in their exponents, may be treated as the following.

Required, the development of $\frac{x}{(1+x)^{\frac{1}{2}}-(1+x)}$.

Let $1+x=z^2$. Then, $\frac{x}{(1+x)^{\frac{1}{2}}-(1+x)}=\frac{z^2-1}{z-z^2}$, which, when developed, gives $z+z^2+z^3+z^4+$ &c. Now, replace z by its value, and we have $\frac{x}{(1+x)^{\frac{1}{2}}-(1+x)}=(1+x)^{\frac{1}{2}}+(1+x)^1+(1+x)^{\frac{3}{2}}+(1+x)^2+$ &c.

RULE.

Place the common parenthesis equal to a new variable, raised to a power denoted by the least common multiple of the denominators of the parentheses, and proceed as before.

EXAMPLES.

1. Expand $\frac{-x}{(1-x)^{\frac{1}{2}}-(1-x)^{-\frac{1}{4}}}$ into a series.

Ans. $(1-x)^{\frac{1}{2}}+(1-x)^{\frac{1}{4}}+(1-x)^0+(1-x)^{-\frac{1}{4}}+$&c.

2. Expand $\frac{-x}{(1-x)^{\frac{1}{3}}+(1-x)^{\frac{2}{3}}}$ into a series.

Ans. $(1-x)^{\frac{2}{3}}-(1-x)+(1-x)^{-\frac{4}{3}}-(1-x)^{-\frac{5}{3}}+(1-x)^{-2}-$&c.

Expressions of the form, $\left(\frac{a-bx}{a+bx}\right)^{\frac{m}{n}}$, may be expanded by placing $a-bx=z^n$, deducing the value of $a+bx$, and proceeding as before.

Take, $\left(\frac{1-x}{1+x}\right)^{\frac{2}{3}}$. Place, $1-x=z^3$. Then, $x=1-z^3$, and $1+x=2-z^3$. Hence, $\left(\frac{1-x}{1+x}\right)^{\frac{2}{3}}=\frac{z^6}{4-4z^3+z^6}=\frac{z^6}{z^6-4z^3+4}$. Develop this fraction and replace z by its value $(1-x)^{\frac{1}{3}}$.

360. It is obvious that there are an infinite number of irrational expressions that may be developed by first making them rational in terms of a new variable, expanding the new expression, and replacing the new variable by its value in terms of the old. So, there are an infinite number of expressions, involving negative exponents, that may

be expanded by substituting, for the old variable, a new variable affected with a positive exponent, and proceeding as before. Thus, let it be required to expand $\frac{x^{-2}-x^{-4}}{1-x^{-4}}$. Let $x^{-2}=z$. Then, $x^{-4}=z^2$. Hence, $\frac{x^{-2}-x^{-4}}{1-x^{-4}}=\frac{z-z^2}{1-z^2}=z-z^2+z^3-z^4+z^5-z^6+$ &c., $=x^{-2}-x^{-4}+x^{-6}-x^{-8}+x^{-10}-x^{-12}+$&c.

There are many expressions, however, involving negative and fractional exponents, that cannot be developed by the method of undetermined coefficients.

DERIVATION.

361. When one quantity depends upon another for its value, it is said to be a function of the quantity upon which it depends. The dependent quantity is the function, and that upon which it depends is called the variable. Thus, in the equation, $y=2x$, y is the function and x the variable, because, by changing the value of x, y may be made to have any system of values. Thus, when $x=0$, $\frac{1}{2}$, 1, 2, 3, &c; $y=0, 1, 2, 4, 6$, &c., it is evident that, in every single equation involving two unknown quantities, there is a function and variable. Either of the unknown quantities may be assumed at pleasure as the function, because both unknown quantities are mutually dependent upon each other. But it is usual to regard that unknown quantity as the function, with reference to which the equation has been solved. If the equation, $y=2x$, is solved with respect to x, we have $x=\frac{y}{2}$. Then, x is the function and y the variable, and, by assuming arbitrary values for y, x may be made to have an infinite system of values. The most general form of a solved equation of the first degree with two unknown quantities is, $y=ax+b$. In this, a and b are called constants, because their values undergo no change. It is plain that y depends for its value upon x, a, and b, but, as its value only changes with x (since x is the only variable), y is called a function of x only. If we have a single equation involving three unknown quantities, any one of these unknown quantities is a function of the other two.

If we have $y=ax$, or $y=ax+b$, and $x=mz$, or $x=mz+n$, we call y an *explicit* function of x, and an *implicit* function of z, because y directly depends upon x for its value, and indirectly upon z. It is plain that we may have explicit functions of any number of variables, and, also, implicit functions of any number of variables. We will, however, confine ourselves to explicit functions of single variables.

If we take the equation, $y = ax + b$, and attribute to x arbitrary values, 0, 1, 2, 3, 4, &c., and call the corresponding values of y, y, y', y'', y''', &c., we will have $y = b$, $y' = a + b$, $y'' = 2a + b$, $y'' = 3a + b$. Then, $y' - y = a$, $y'' - y' = a$, $y''' - y'' = a$, &c. That is, the difference between any two consecutive states of the function is constant. This constant increase of the state of the function is called the differential or derivative of the function. It is evident that, when the function and variable are both linear, that is, both of the first degree, the difference between two consecutive states of the function will always be constant, and then will be truly the derivative of the function. Thus, take the equation, $y = 2x + 2$, and let x have the constant increment $\frac{1}{2}$, beginning at 0; that is, let $x = 0$, $0 + \frac{1}{2}$, $\frac{1}{2} + \frac{1}{2}$, $1 + \frac{1}{2}$, $1\frac{1}{2} + \frac{1}{2}$; then the difference between two consecutive values of y will always be constant. For, we have $y = 2$, $y' = 1 + 2 = 3$, $y'' = 2 + 2 = 4$, $y''' = 3 + 2 = 5$, &c., and, consequently, $y' - y = 1 = y'' - y' = y''' - y''$, &c.

But, when the variable is of the second degree, and the function of the first, the difference between two consecutive states of the function, corresponding to constant increments of the variable, will not be constant. Thus, take the equation, $y = x^2$, and let x have the constant increment, 1, beginning at zero. Then, $y = 0$, $y' = 1$, $y'' = 4$, $y''' = 9$, &c., and $y' - y = 1$, $y'' - y' = 3$, $y''' - y'' = 5$. The student of geometry will understand that there ought to be no constant increment to the function, corresponding to a constant increment to the variable in the equation, $y = x^2$. For, y expresses the surface of a square of which x is the side, and the surface of course increases more rapidly than the side. If the function is of the first degree, and the variable of the third, it will be seen that the difference between two consecutive states of the function varies still more widely from a constant. The function and variable must then both be of the first degree, in order that the difference between two consecutive states of the function may be constant. But, to return to the illustration of the square, it is plain that, if the constant increment to the side of the square was indefinitely small, the difference between two consecutive states of the function, which represents the increment of the function, would be so nearly constant that it might be regarded as constant. Suppose $y = x^2$, and that $x = 1$ foot. Then the surface of the square is one square foot. Now, suppose the side of the square to receive the constant increment, $\frac{1}{10000}$ part of a foot. Then, $y = 1, y' = 1 + \frac{2}{10000} + \frac{1}{(10000)^2}$, $y'' = 1 + \frac{4}{10000} + \frac{4}{(10000)^2}$, &c. Then, $y' - y = \frac{2}{10000} + \frac{1}{(10000)^2}$, $y'' - y = \frac{2}{10000} + \frac{2}{(10000)^2}$. And we see that the difference between the second and first states differs only from the difference between the third and second, by $\frac{1}{10000}$.

Upon this principle the derivative or increment of a function is found. It is the difference between two consecutive states of the function, when these states are indefinitely near to each other, and it expresses the increment of the function. The word increment is used in its algebraic sense. When any state of the function is less than the preceding, the derivative is truly a decrement.

To find the derivative of the function, $y = x^2$, let h represent the infinitely small constant increment to x. Then, $y = x^2$, $y' = (x + h)^2 = x^2 + 2xh + h^2$, and $y' - y = 2xh + h^2$. Now, since h is infinitely small by hypothesis, h^2 will be an infinitely small quantity of the second order, and may therefore be neglected. Thus, $\frac{1}{1 \text{ million}}$ is very small, but $\frac{1}{(1 \text{ million})^2}$ is much smaller. Now, by making h indefinitely small, we have taken the states indefinitely near to each other, and, consequently, $y' - y$ is the derivative of $y = x^2$. Hence, the derivative of x^2 is $2xh$. It is usual to represent h the increment of the function by dx, read differential of x. The derivative of the square of any variable function is then twice the first power of the variable into the differential of the variable. The differential coefficient of a function is the differential of the function divided by the increment of the variable, and is generally expressed by $\frac{y' - y}{h}$. In the preceding example the differential coefficient of x^2 is $2x$; and, in general, knowing the differential of the function, we get the differential coefficient by dividing by the increment or differential of the variable; and, conversely, we get the differential of the function from the differential coefficient, by multiplying by the differential of the variable. It will be seen that, when

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we make $h = 0$, or indefinitely small, that the differential coefficient assumes the form of $\frac{0}{0}$, for then $y' = y$.

To find the differential coefficient, we have the following

RULE.

Give to the variable a variable increment, h, *and find the new state of the function. Take the difference between the new and old states of the function, and reject the terms involving the higher powers of the increment, as being infinitely small quantities of the second, third, &c., orders. Next change* h *into* dx, dy, *or* dz, *&c., according as the variable is* x, y, *or* z, *&c. We then have the differential of the function. Divide the differential of the function by the differential of the variable, and we have the differential coefficient.*

Required the differential coefficient of the function, $y = x^3$. By the rule, we have $y' = (x + h)^3 = x^3 + 3x^2h + 3h^2x + h^3$. Hence, $y' - y = 3x^2h + 3h^2x + h^3 = 3x^2h$, or $3x^2dx$, when h is infinitely small. Therefore, $\frac{y' - y}{h} = 3x^2$.

Newton regarded all algebraic expressions as the representatives of lines, surfaces, or solids; and supposed lines, whether straight or curved, to be generated by the flowing of points according to fixed laws, surfaces to be generated by the flowing of lines, and solids to be generated by the flowing of surfaces. Thus, a point, moving or flowing according to the law that it shall always be in the same plane, and at the same distance from a point, will generate the circumference of a circle; and a straight line, flowing with the two conditions of being constantly parallel to, and at the same distance from a fixed line, will generate the surface of a cylinder. A straight line flowing in the same direction with the condition that, in all its positions it shall continue parallel to itself in its first position, will generate a plane. One line revolving around another, to which it is perpendicular, $\perp$, will generate the surface of a circle. A flowing square will generate a cube; a flowing semicircle, a sphere. Newton regarded the increment of the function as expressing the uniform rate of increase of the function, and called it *fluxion.* The thing generated he called *fluent.* It is evident that the fluxion does not express the uniform rate of increase, except when the states are taken indefinitely near to each other. Suppose a cannon-ball to leave the mouth of a piece with a velocity of 2000 feet per second, and that this velocity is reduced to 1200 feet per second at the end of the third second. It is plain, that the difference between the spaces passed over in any two consecutive instants of time will not be equal to the difference between the distances passed over in any other two consecutive instants, unless those instants are inappreciably small. But, for the millionth part of a second, the velocity might be regarded as constant. If the instant was then taken thus small, the difference between the spaces in two consecutive instants would be constant.

The differential of the function has, in accordance with the Newtonian theory, been defined to be the uniform rate of increase or decrease of the function. The differential of a constant then, must be zero, since a constant admits of neither increase or decrease.

Theorem I.

362. The differential of the algebraic sum of any number of functions of the same variable is equal to the algebraic sum of their differentials.

Let $y = \pm u \pm v \pm w \pm z$; u, v, w, and z, being functions of the same variable, x. Give to the variable a variable increment, h, and call the new states of the function u', v', w', and z'. Then $y' = u' \pm v' \pm w' \pm z'$, and $y' - y = u' - u \pm (v' - v) \pm (w' - w) \pm (z' - z)$ as ennunciated, since $u' - u$ is the differential of u, $v' - v$, the differential of v. A shorter notation is $dy = du \pm dv \pm dw \pm dz$, and is read differential of y, equal to the differential of u, plus or minus the differential of v, &c.

Let $y = ax^2 + cx$. Then, $dy = 2axdx + cdx$.

Theorem II.

363. The differential of the product of a constant by a variable, is equal to the product of the constant by the differential of the variable.

Let $y = Ax$. Then, $dy = Adx$. For, $y' = A(x + h)$, and $y' - y = Ah$, or $dy = Adx$.

Let $u = my$. Then, $du = mdy$.

Theorem III.

364. The differential coefficient of the power of a quantity is equal to the exponent of the power into the quantity affected with an exponent less by unity than the primitive exponent.

Let $y = x^n$. Then, $\frac{y' - y}{h} = nx^{n-1}$.

For, give to x a variable increment, h; then, $y' = (x + h)^n$, and $\frac{y' - y}{h} = \frac{(x + h)^n - x^n}{h} = \frac{(x + h)^n - x^n}{(x + h) - x} = \frac{a^n - x^n}{a - x}$, by representing $x + h$ by a. But, $\frac{a^n - x^n}{a - x} = a^{n-1} + a^{n-2}x + a^{n-3}x^2 + a^{n-4}x^3 +$ &c., on to n terms. Now, when h is made indefinitely small, x is equal to a, the first member of the equation in y then becomes equal to the differential coefficient, and the second member becomes $a^{n-1} + a^{n-1} + a^{n-1} +$ &c. $= na^{n-1}$, as enunciated.

Hence, we have $\frac{dy}{dx} = na^{n-1}$, and, consequently, $dy = na^{n-1}dx$. The differential, as well as the differential coefficient of a power, is then known.

Theorem IV.

365. The differential of the product of two functions of the same variable is equal to the sum of the products which arise from multiplying each function by the differential of the other function.

Let $y = uv$. Then $dy = udv + vdu$, in which u and v are both functions of the same variable, x; the new state of the function is $u' = u + du$, and the new state of the function $v' = v + dv$. Hence, $y' = u'v' = (u + du)\ (v + dv) = uv + udv + vdu + dudv$. But, since du and dv are indefinitely small, their product, $dudv$, is an indefinitely small quantity of the second order, and may therefore be neglected. Hence, $y' = uv + udv + vdu$, and $y' - y$, or $dy = udv + vdu$, as enunciated.

Corollary.

366. 1st. The differential of the product of any number of functions of the same variable is equal to the sum of the products which arise from multiplying the differential of each function by the product of the other functions.

Let $y = swuz$, in which s, w, u, and z are functions of the same variable, z. Then, $dy = wuzds + suzdw + swzdu + swudz$.

To show this, we will first get the differential of the product of three variables. Let $y = szv$. Make $sz = u$. Then $y = uv$, and, from what has just been shown, $dy = udv + vdu = szdv + vd\ (sz) = szdv + v\,(sdz + zds) = szdv + vsdz + vzds$, as enunciated.

Now, knowing the differential of the product of three variable functions, we can readily pass to that of four. For, let $y = swuz$. Then, $dy = swudz + zd(swu) = swudz + z(swdu + sudw + wuds) = swudz + zswdu + zsudw + zwuds$. And, it is plain that the same process can be extended to the product of any number of functions.

2d. Since the differential of the product of Av is $Adv + vdA = Adv$, A being a constant, we have Theorem II. demonstrated in another way.

THEOREM V.

367. The differential of a fraction is the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.

Let $y = \frac{s}{v}$, s and v being both functions of x. Then, $dy = \frac{vds - sdv}{v^2}$.

For, since $y = \frac{s}{v}$, we have $yv = s$; and yv being equal to s, the increment of yv must be equal to the increment of s. That is, $d(yv) = ds$, or $ydv + vdy = ds$, or $dy = \frac{ds}{v} - \frac{ydv}{v} = \frac{ds}{v} - \frac{sdv}{v^2} = \frac{vds - sdv}{v^2}$, as enunciated. In the expression, $\frac{ydv}{v}$, we substituted for y its value, $\frac{s}{v}$, and then reduced the two fractions to a common denominator.

Corollary.

368. 1st. When the denominator is constant, dv is zero, and $dy = \frac{vds - sdv}{v^2} = \frac{vds}{v^2} = \frac{ds}{v}$, equal to the differential of the numerator divided by the constant denominator. We might arrive at this result in another way, for, when $v = a$ constant, $\frac{s}{v}$ may be written $\frac{1s}{v}$, and, since $\frac{1}{v}$ is constant, the differential of $\frac{1s}{v}$ will, by Theorem II., be $\frac{1ds}{v}$, or $\frac{ds}{dv}$.

369. 2d. When the numerator is constant, $ds = 0$, and $dy = \frac{vds - sdv}{v^2}$, becomes $dy = -\frac{sdv}{v^2}$, a negative result.

This ought to be so, for, when the numerator is constant and the denominator variable, any increment to the variable, x, upon which v depends, will decrease y; and, consequently, dy, which expresses the algebraic increment of y, is truly a decrement, and ought, therefore, to have the negative sign. In the case supposed, y is a decreasing function of the variable, x, and we see that *the differential of a decreasing function is negative.*

We are now prepared to find the differential of any function affected with any exponent, whether positive or negative, fractional or entire.

Theorem VI.

370. The differential of a quantity affected with any exponent, is the product arising from multiplying the exponent of the power into the quantity affected with an exponent algebraically less by unity than the primitive exponent, into the differential of the variable.

We will first suppose the quantity to be affected with a negative exponent. Let $y = v^{-n} = \frac{1}{v^n}$. Then, by Corollary 2d, of the last Theorem, $dy = -\frac{1d(v^n)}{v^{2n}} = -\frac{nv^{n-1}}{v^{2n}} = -nv^{-n-1}$, as enunciated.

Next, suppose the exponent to be a positive fraction, $y = v^{\frac{r}{s}}$. Then, $y = \sqrt[s]{v^r}$, or $y^s = v^r$. And, taking the increments of both members, we have, by Theorem III., since r and s are positive and entire, $sy^{s-1}dy = rv^{r-1}dv$, or $dy = \frac{rv^{r-1}dv}{sy^{s-1}} = \frac{r}{s} \times \frac{v^{r-1}dv}{y^{s-1}} = \frac{r}{s} \times \frac{v^{r-1}dv}{(v^{\frac{r}{s}})^{s-1}}$

$= \frac{r}{s} \times \frac{v^{r-1}dv}{v^{\frac{rs-r}{s}}} = \frac{r}{s} \times \frac{v^{\frac{rs-s}{s}}dv}{v^{\frac{rs-r}{s}}} = \frac{r}{s} v^{\frac{r-s}{s}}dv = \frac{r}{s} v^{\frac{r}{s}-1}dv$, as enunciated.

Finally, suppose $y = v^{-\frac{r}{s}}$; we will find by proceeding exactly in the same way as when the exponent is a positive fraction, $dy = -\frac{r}{s} \times v^{-\frac{r}{s}-1}dv$, as enunciated.

Corollary.

370. 1st. The differential of any parenthetical expression is equal to the exponent of the parenthesis into the parenthesis, raised to a power algebraically less by unity than at first, into the differential of the quantity within the parenthesis.

This is evident, since we may represent the quantity within the parenthesis by a single variable, v. The parenthetical expression will then assume the form of v^n, and may be differential according to the rule.

Let $y = (a + bx^2)^n$, place $a + bx^2 = v$, then $y = v^n$, and $dy = nv^{n-1}dv = n(a + bx^2)^{n-1}d(a + bx^2) = n(a + bx^2)^{n-1}\ 2bxdx = 2bn(a + bx^2)^{n-1}xdx$.

372. 2d The differential of a radical expression is equal to the differential of the quantity under the radical, divided by the index of the radical into the radical raised to a power algebraically less by unity than the primitive index. This is merely a particular case of the preceding, for a radical is nothing more than a parenthetical expression affected with a fractional exponent, the numerator of the fraction being unity.

Let $\sqrt[n]{v}$ be the radical; this is equal to $v^{\frac{1}{n}}$, and its differential then must be $\frac{1}{n}v^{\frac{1}{n}-1}dv = \frac{1}{n}v^{\frac{1-n}{n}}dv = \frac{dv}{nv^{\frac{n-1}{n}}} = \frac{dv}{n\sqrt[n]{v^{n-1}}}$, as enunciated. When $n = 2$, the radical is of the second degree, and the expression becomes $\frac{dv}{2\sqrt{v}}$, that is, the differential of a radical of the second degree is equal to the differential of the quantity under the radical sign divided by twice the radical.

EXAMPLES.

1. Required the derivative of $y = \sqrt[3]{a + x^2}$.

Ans. $dy = \frac{2xdx}{3\sqrt[3]{(a + x^2)^2}}$.

2. Required the derivative of $y = \sqrt{a + bx^2}$.

Ans. $dy = \frac{2bxdx}{2\sqrt{a + bx^2}}$.

3. Required the derivative of $\sqrt[m]{ax + bx^n}$.

Ans. $\frac{(a + nbx^{n-1})\,dx}{m\sqrt[m]{(ax + bx^n)^{m-1}}}$.

GENERAL EXAMPLES.

1. Required the derivative of $a + bx - cx^2 + mx^3$.

Ans. $(b - 2cx + 3mx^2)\,dx$.

2. Required the derivative of $\dfrac{a + bx}{cx^2}$.

Ans. $\dfrac{cx^2bdx - (a + bx)\,2cxdx}{c^2x^4}$.

3. Required the derivative of $x^m(a + bx)^n$.

Ans. $m(a + bx)^n x^{m-1}\,dx + nx^m(a + bx)^{n-1}\,bdx$.

BINOMIAL FORMULA.

373. By actual multiplication we can readily find

$$(a + x)^2 = a^2 + 2ax + x^2,$$
$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3,$$
$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4a^3x^3 + x^4.$$

We observe a simple law in regard to the exponents in these three developments. The exponent of a in the first term is the power of the binomial, and this exponent goes on decreasing by unity unto the last term, in which it is zero. For, x^2, x^3, and x^4 may be written, a^0x^2, a^0x^3, and a^0x^4. The exponent of x is zero in the first term, and goes on increasing to the last term, in which it is equal to the power of the binomial. With regard to the coefficients, we observe that the coefficients of the extreme terms are unity, that the coefficient of the second term is the degree of the binomial, and that the coefficient of the third term can be found by multiplying the coefficient of the second term by the exponent of a in that term, and dividing by the number of terms which precede the required term. Thus, in the development of $(a + x)^4$, to find this coefficient, multiply 4, the coefficient of the second term, by 3, the exponent of a in that term, and divide by 2, the number of terms which precede the required term, the quotient 6 is the coefficient of the third term. So, to find the coefficient of the fourth term of $(a + x)^4$, multiply 6, the coefficient of the third term, by 2, the exponent of a in that term, and divide the product, 12, by 3, the number of terms which precede the required term, the quotient, 4, is the coefficient of the fourth term. This remarkable law in regard to the coefficients, holds true even for the second and last terms.

We observe, furthermore, that the sum of the exponents of a and x in every term is equal to the exponents of the binomial and that the coefficients, at equal distances from the extremes, are equal.

Newton showed that the law in regard to the exponents and coefficients was general for any binomial of the form $(a + x)^m$, and that we would have $(a + x)^m = a^m + ma^{m-1}x + \frac{m(m-1)a^{m-2}x^2}{2} + \frac{m(m-1)(m-2)a^{m-3}x^3}{2 \cdot 3} + \ldots\ldots\ldots\ldots x^m$.

DEMONSTRATION.

374. To demonstrate this useful and remarkable Theorem, let us assume $(a + x)^m = A + A'x + A''x^2 + A'''x^3 + \ldots\ldots A^m x^m$; in which m is taken positive and entire, and A', A'', &c., independent of x. Then, by the principle of undetermined coefficients, we have a right to make $x = 0$ in both members of the assumed identical equation. Making $x = 0$, we have $A = a^m$.

It was shown that, when two functions of the same variable were equal, that their differentials were equal. Hence, taking the derivations of both members of the assumed equation, we have $m(a + x)^{m-1}dx = (A' + 2A''x + 3A'''x^2 + \&c.)dx$, or, dividing by dx, $m(a + x)^{m-1} = A' + 2A''x + 3A'''x^2 + \&c.$ (N). Now, make $x = 0$, and we get, $ma^{m-1} = A'$. Again, taking the derivatives of both members of (N), and dividing by dx, we get, $m(m-1)(a + x)^{m-2} = 2A'' + 2 \cdot 3 \cdot A'''x + \&c.$ (P). Making $x = 0$, we have $A'' = \frac{m(m-1)a^{m-2}}{2}$. Again, taking the derivation of (P), dividing by dx, and making $x = 0$, there results $A''' = \frac{m(m-1)(m-2)a^{m-3}}{2 \cdot 3}$. In like manner, we can find $A'''' = \frac{m(m-1)(m-2)(m-3)a^{m-4}}{2 \cdot 3 \cdot 4}$.

And A^m will plainly be equal to

$$\frac{m(m-1)(m-2)\ldots\ldots(m-(m-1))a^{m-m}}{1 \cdot 2 \cdot 3 \ldots\ldots (m-2)(m-1)m};$$

and, since the factors of the denominator are the same as those of the numerator, only written in reverse order, we have $A^m = 1$.

Now, replace A, A′, A″, &c., by their values in the assumed identical equation, and we have $(a + x)^m = a^m + ma^{m-1}x + \frac{m(m-1)a^{m-2}x^2}{1 \cdot 2} + \frac{m(m-1)(m-2)a^{m-3}x^3}{1 \cdot 2 \cdot 3} + \ldots\ldots\ldots$

$$\frac{m(m-1)(m-2)\ldots(m-n+2)a^{m-n+1}x^{n-1}}{1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1)} + \ldots\ldots x^m.$$

The term,

$$\frac{m(m-1)(m-2)\ldots(m-n+2)a^{m-n+1}x^{n-1}}{1 \cdot 2 \cdot 3 \ldots (n-1)},$$

is called the n^{th}, or general term. An inspection of the formula will show that the first factor of the numerator of the coefficient of a in any term is m, the next factor $(m-1)$, and so, on decreasing by unity, unto the last factor, which is m, diminished by two less than the place of the term. Thus, the last factor of the numerator of the coefficient of the 4th term is $(m-2)$. So, the last factor of the numerator of the coefficient of the n^{th} term must be $(m-n+2)$. The denominator of the coefficient of a, in any term, is always the continued product of the natural numbers, from 1 up to one less than the place of the term. Hence, the denominator of this coefficient in the n^{th} term must be $1 \cdot 2 \cdot 3 \ldots\ldots (n-1)$. In regard to the exponents, we see that a, in any term, is always affected with the exponent of the binomial, diminished by the number of the term less one, and the exponent of x, in any term, is always one less than the place of the term. Hence, for the n^{th} term, we have $a^{m-n+1}x^{n-1}$.

The binomial formula is usually written in the ascending powers of a, instead of x. To develop $(x + a)^m$ in the powers of a, it will be necessary to regard a as the variable, and x as the constant, to make $a = 0$ in the successive steps of the operation. As we would obviously obtain the same result, with the exception of an interchange of x and a, we may at once write

$$(x + a)^m = x^m + mx^{m-1}a + \frac{m(m-1)x^{m-2}a^2}{1 \cdot 2} +$$

$$\frac{m(m-1)(m-2)x^{m-3}a^3}{1 \cdot 2 \cdot 3} + \ldots \frac{m(m-1)\ldots(m-n+2)x^{m-n+1}a^{n-1}}{1 \cdot 2 \cdot 3 \ldots (n-1)}$$

$$\ldots\ldots a^m.$$

We will repeat the laws in regard to the exponents and coefficients.

1. The first letter of the binomial appears as the first term of the

development, affected with an exponent equal to that of the binomial, and the exponents of this letter in the successive terms decrease by unity unto the last term, where its exponent is zero. The exponent of the second letter of the binomial is zero in the first term of the development, and one greater in each of the successive terms, and, in the last term is m, the exponent of the binomial.

2. The sum of the exponents of x and a, in every term of the development, is equal to the exponent of the binomial. A simple inspection of the formula will show this.

3. The coefficient of the first term of the development is unity; that of the second term is equal to the exponent of the binomial; that of the third term is formed by multiplying the exponent of the first letter of the second term by the coefficient of that letter, and dividing by one less than the number denoting the place of this required term; that of the n^{th} term is formed from the $(n-1)^{th}$ term, by multiplying together the coefficient and exponent of the first letter of that term, and dividing this product by $(n-1)$, that is, by one less than the number of the required term.

4. There will always be $m + 1$ terms in the development. For, we get one term, a^m, or x^m, without differentiating, and, since the exponent of m is diminished by unity for each differentiation, it is plain that m derivatives can be taken of $(a + x)^m$, and, since we get a term of the development by each derivation, we must have in all $(m + 1)$ terms.

5. The coefficients at equal distances from the extremes are equal. For the developments of $(a + x)^m$ and $(x + a)^m$ must be identical, differing only in the order of the terms; the first term of the development of $(a + x)^m$ being the last term of that of $(x + a)^m$, the second from the left of the development of $(a + x)^m$ being the second from the right of that of $(x + a)^m$, &c. Hence, the terms of the development of $(a + x)^m$, taken in reverse order, will constitute the direct development of $(x + a)^m$. It is plain, then, that the second term from the left must have the same coefficient as the second term from the right, the one being formed from x^m in the same manner as the other from a^m, and so, in like manner, all the other coefficients at equal distances from the extremes must be equal.

6. The sum of the coefficients of the development of $(x + a)^m$ is equal to the m^{th} power of 2. For, if we make $x = 1$ and $a = 1$ in the equation, $(x + a)^m = x^m + mx^{m-1}a + \frac{m(m-1)x^{m-2}a^2}{1 \cdot 2} +$

$\frac{m\,(m-\ )\,(m-2)\,x^{m-3}\,a^3}{1\,.\,2\,.\,3}$ .... a^m, the second member will reduce to the sum of the coefficients, and the first member will become 2^m, and we will have $(1+1)^m = 2^m = 1 + m + m\,(m-1) + \frac{m\,(m-1)\,(m-2)}{1\,.\,2}$ + &c. We make x and a unity, in order to reduce the terms in the second member to their coefficients; and, it is plain, that if x and a had another value than unity, the second member would not be the sum of the coefficients.

In accordance with the demonstration, the sum of the coefficients in the development $(x+a)^2$, ought to be 4; for, in this case, $m = 2$, and $(2)^m = (2)^2 = 4$. And, in the development of $(x+a)^3$, $m = 3$, and $(2)^m = (2)^3 = 8$, which agrees with the fact. So, likewise, in the development of $(x+a)^4$, $m = 4$, and $(2)^m = (2)^4 = 16$, which also agrees with the fact.

FORMATION OF POWERS BY THE RULE.

375. Let it be required to find the development of $(x+a)^5$ by the rule. The first term is x^5, the second term has 5 for a coefficient, and the exponent of x is one less in this term than in the preceding; a also enters to the zero power in the first term; and, since the exponents of a go on increasing by unity, a must enter to the first power in the second term. Hence, the second term is $5x^4a$. The coefficient of x in the third term is formed by multiplying 5, the coefficient of x in the second term, by 4, its exponent, and dividing the product by 2, the number of terms preceding the required term. Hence, $\frac{5\times 4}{2} = 10$, is the coefficient of the third term, and, from the law of the exponents, x must enter to the third power, and a to the second power in the third term. Hence, that term is $10x^3a^2$. For the coefficient of x in the fourth term, we have $\frac{10\,.\,3}{3} = 10$, and the fourth term must be $10x^2a^3$. For the coefficient of the fifth term we have $\frac{10\,.\,2}{4} = 5$, and that term itself must be $5xa^4$. For the coefficient of the sixth term we have $\frac{5\,(1)}{5} = 1$, and that term itself must be x^0a^5, or a^5. For the coefficient of the seventh term we have $\frac{1\,.\,0}{6} = 0$. Hence, there is no

seventh term. Then, the development of $(x + a)^5$ is $x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$, and we see that the sum of the coefficients is equal to $(2)^5 = 32$, and that the sum of the exponents of x and a in each term is 5.

The preceding development might have been obtained directly from the general formula, $(x+a)^m = x^m + mx^{m-1}a + \frac{m(m-1)x^{m-2}a^2}{1\ .\ 2} + \&c.$, by making $m = 5$. But the above process is generally shorter when m is positive and entire.

DEVELOPMENT OF $(x - a)^m$.

376. The development of $(x - a)^m$ may be obtained in the same way as that of $(x + a)^m$. But it may be gotten at once from the general formula by changing $+a$ into $-a$; the terms involving the odd powers of a will all be negative, and we will have $(x - a)^m = x^m - mx^{m-1}a + \frac{m(m-1)x^{m-2}a^2}{1\ .\ 2} - \frac{m(m-1)x^{m-3}a^3}{1\ .\ 2\ .\ 3} + \ldots\ldots\ldots \pm a^m$. The last term will be positive when m is an even number, and negative when m is an odd number.

It will be seen that all the laws in regard to the development of $(x + a)^m$ hold good in regard to the development of $(x - a)^m$, except that the alternate terms must be affected with the negative sign, and that the sum of the coefficients is zero.

A few examples will illustrate the use of the two formulas for the development of $(x + a)^m$ and $(x - a)^m$.

EXAMPLES.

1. Required the sixth power of $(x + a)$.

Ans. $x^6 + 6x^5a + 15x^4a^2 + 20x^3a^3 + 15x^2a^4 + 6xa^5 + a^6$.

For, $m = 6$, $m - 1 = 5$, $(m - 2) = 4$, $m - 3 = 3$, $m - 4 = 2$, $m - 5 = 1$. Hence, $(x + a)^m = x^m + mx^{m-1} + \frac{m(m-1)x^{m-2}a^2}{2} + \ldots\ a^m$ becomes, in this case, $(x+a)^6 = x^6 + 6x^5 + \frac{6\ .\ 5x^{6-2}a^2}{2} \ldots\ldots a^6 = x^6 + 6x^5 + 15x^4a^2 + \ldots\ldots\ a^6$.

2. Required the development of $(x + a)^7$.

Ans. $x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7$.

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3. Develop $(x+a)^8$.

Ans. $x^8+8x^7a+28x^6a^2+56x^5a^3+70x^4a^4+56x^3a^5+28x^2a^6+8xa^7+a^8$.

4. Develop $(x-a)^6$.

Ans. $x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6$.

5. Develop $(x-a)^7$.

Ans. $x^7-7x^6a+21x^5a^2-35x^4a^3+35x^3a^4-21x^2a+7xa^6-a^7$.

6. Required the development of $(x-a)^9$.

Ans. $x^9-9x^8a+36x^7a^2-84x^6a^3+126x^5a^4-126x^4a^5+84x^3a^6-36x^2a^7+9xa^8+a^9$.

377. The binomial formula has been deduced upon the hypothesis that the coefficients and exponents of x and a were unity, but it can be applied to binomials in which these conditions are not fulfilled, by substituting for the letters within the parenthesis other letters whose coefficients and exponents are unity. Thus, to get the fifth power of z^2+3y^2, let $z^2=x$, and $3y^2=a$; then, $(z^2+3y^2)^5=(x+a)^5=x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$, which, by substituting for x and a their values, will be found equal $z^{10}+15z^8y^2+90z^6y^4+270z^4y^4+405z^2y^8+243y^{10}$.

In like manner, we may find the development of $(nx^p+ra^q)^m=$

$$n^mx^{pm}+mrn^{m-1}x^{pm-p}a^q+\frac{m\,(m-1)\,r^2n^{m-2}x^{pm-2p}a^{2q}}{2}+\ldots r^ma^{qm}.\quad\text{(A)}$$

In this general formula, make $x=1$, and $a=1$; the second member will reduce to the sum of the coefficients of x, and the first member will become $(n+r)^m$. *Hence, the sum of the coefficients of any binomial development is equal to the* m[th] *power of the sum of the coefficients within the parenthesis.*

In accordance with this general formula, the sum of the coefficients of the development of $(z^2+3y^2)^5$ ought to be equal to $(4)^5=1024$; and this agrees with the development above, for $1+15+90+270+405+243=1024$.

378. Formula (A) shows, moreover, that if the exponents of x and a are equal, that is, $p=q$, the sum of the exponents of x and a in each term of the development will pm. *Whenever, then, the exponents of* x *and* a *are equal in any binomial, the sum of the exponents of these letters in each term of the development will be equal to the common exponent within the parenthesis into the exponent of the power to which the binomial is to be raised.*

The two important laws just deduced from formula (A) are general for all binomials. The second and sixth laws, observed to govern the development of $(x + a)^m$, are but particular cases of the foregoing.

The binomial formula can be extended to polynomials, by representing the polynomial by the algebraic sum of two letters. Thus, to obtain the square of $x^3 + 4x^2 - 8x - 8$, represent $x^3 + 4x^2$ by x, and $-8x - 8$, by a. Then, $(x^3 + 4x^2 - 8x - 8)^2 = (x + a)^2 = x^2 + 2ax + a^2 =$ (by replacing x and a by their values), $x^6 + 8x^5 - 80x^3 + 128x + 64$. In like manner, to obtain the cube of $x^2 + 2x - 4$, let $x^2 = x$, and $2x - 4 = a$; then, $(x^2 + 2x - 4)^3 = (x + a)^3 = x^3 + 3x^2a + 3a^2x + a^3$, which, by replacing x and a by their values, becomes $x^6 + 6x^5 - 40x^3 + 96x - 64$.

GENERAL EXAMPLES.

1. Find the coefficient of the twelfth term of the development of $(x + a)^m$, by means of the formula for the n^{th} term.

Ans. $\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)(m-7)(m-8)(m-9)(m-10)}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11}$.

2. Find the twelfth term of the development of $(x + a)^{50}$.

Ans. $\frac{50\times49\times48\times47\times46\times45\times44\times43\times42\times41\times40a^{39}x''}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11}$.

3. Find the development of $(x + a)^{20}$.

Ans $x^{20} + 20x^{19}a + 190x^{18}a^2 + 1140x^{17}a^3 + 4845x^{16}a^4 + 15504x^{15}a^5 + 38760x^{14}a^6 + 77520x^{13}a^7 + 125970x^{12}a^8 + 167960x''a^9 + 184756x^{10}a^{10} + 167960x^9a'' + 125970x^8a^{12} + 77520x^7a^{13} + 38760x^6a^{14} + 15504x^5a^{15} + 4845x^4a^{16} + 1140x^3a^{17} + 190x^2a^{18} + 20xa^{19} + a^{20}$.

4. Find the development of $(x^2 + 2ax - 4a^2)^3$.

Ans. $x^6 + 6ax^5 - 40a^3x^3 + 96a^5x - 64a^6$.

5. Find the 4th power of $3a^2c - 2bd$.

Ans. $81a^8c^4 - 216a^6c^3bd + 216a^4c^2b^2d^2 - 96a^2cb^3d^3 + 16b^4d^4$.

6. Find the cube of $x^2 - 2x + 1$

Ans. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

7. Required the coefficient of the sixth term of the development of $(x + a)^7$.

Ans. $\frac{7(7-1)(7-2)(7-3)(7-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5} = \frac{7\,.\times6\,.\,5\times4\times3}{1\,.\,2\,.\,3\,.\,4\,.\,5} = 21$.

DEMONSTRATION OF THE BINOMIAL FORMULA FOR ANY EXPONENT.

379. 1. Let m be supposed negative and entire, then $(x + a)^{-m} = \frac{1}{(x+a)^m} = \frac{1}{x^m + mx^{m-1}a + \frac{m(m-1)x^{m-2}a^2}{1\,.\,2} + \ldots a^m}$, which, by actually performing the division, will be found equal to $x^{-m} - mx^{-m-1}a - \frac{m(-m-1)x^{-m-2}a^2}{1\,.\,2} - \frac{m(-m-1)\,(-m-2)x^{-m-3}a^3}{1\,.\,2\,.\,3} \ldots \pm Px^{-2m}a^m$, &c. Hence, we have the same law of formation as when the exponent is positive and entire. We might demonstrate this truth more rigorously by assuming $(x + a)^{-m} = A + A'a + A''a^2 + A'''a^3$, &c., and proceeding just as we did when m was positive and entire, regarding a as the variable. But it is not necessary to repeat the operation. We have a right to assume the exponents of a to be positive and entire in all the terms of the development; because, if we expand any fraction whose numerator is a constant, and the first term of whose denominator is a constant, all the exponents of the variable will be positive and entire in the development.

2. When m is positive and fractional. Let $m = \frac{p}{q}$, and assume $(a + x)^{\frac{p}{q}} = A + A'x + A''x^2 + A'''x^4 \ldots\ldots A^nx^n$(P). Make $x = 0$, and we have $A = a^{\frac{p}{q}}$. Taking the derivatives of both members of (P), and dividing by dx, we have $\frac{p}{q}(a + x)^{\frac{p}{q}-1} = A' + 2A''x + 3A'''x^2 +$ &c. (2). Making $x = 0$, we get $A' = \frac{p}{q}a^{\frac{p}{q}-1}$. From (2), there results $\frac{p}{q}\left(\frac{p}{q} - 1\right)(a + x)^{\frac{p}{q}-2} = 2A'' + 2\,.\,3A'''x +$ &c. (R). Making $x = 0$, we get $A'' = \frac{p}{q}\left(\frac{p}{q} - 1\right)\frac{a^{\frac{p}{q}-1}}{1\,.\,2}$. From (R), there results $\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)(a + x)^{\frac{p}{q}-3} = 2\,.\,3A''' +$ &c. From which, $A''' = \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)a^{\frac{p}{q}-3}}{2\,.\,3}$

Continuing the process and replacing the constants, A, A′, &c., by their values, equation (P) becomes $(a+x)^{\frac{p}{q}} = a^{\frac{p}{q}} + \frac{p}{q}a^{\frac{p}{q}-1}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)a^{\frac{p}{q}-2}x^2}{1\,.\,2} + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)a^{\frac{p}{q}-3}x^3}{1\,.\,2\,.\,3} \ldots\ldots \pm \mathrm{P}a^{\frac{p}{q}-(p-1)}x^{p-1}$.

DEVELOPMENT OF BINOMIALS AFFECTED WITH NEGATIVE AND FRACTIONAL EXPONENTS.

380. When m, the exponent of the parenthesis, $(x+a)^m$, is positive and entire, it is evident that successive differentiation will eventually reduce that exponent to zero, and then the next differential will be zero. And, since the constants, A, A′, &c., of the assumed development, have been determined by the successive differentiation of $(x+a)^m$, it is plain that the development will terminate. But, when the exponent of the parenthesis is negative or fractional, successive differentiation can never reduce it to zero, and, therefore, an infinite number of constants can be determined in the equation, $(x+a)^{-m}$, or, $(x+a)^{\frac{p}{q}}.\,. = \mathrm{A} + \mathrm{A}'x + \mathrm{A}''x^2 + \mathrm{A}'''x^3 + \&c.$ The series will then never terminate.

EXAMPLES.

1. Develop $\frac{1}{1-x} = (1-x)^{-1}$ into a series.

Ans. $1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$

Make $m = -1$, $1 = x$, and $-x = a$ in the formula $(x+a)^{-m} = x^{-m} - mx^{-m-1}a - \frac{m(-m-1)x^{-m-2}a^2}{1\,.\,2} - \&c.$

2. Develop $\frac{1}{1+x}$ into a series.

Ans. $1 - x + x^2 - x^3 + x^4 - x^5 + \&c.$

3. Develop $\frac{1+7x+x^2}{1+x} = (1+7x+x^2)(1+x)^{-1}$ into a series.

Ans. $1 + 6x - 5x^2 + 5x^3 - 5x^4 + 5x^5 - \&c.$

4. Develop $\frac{a}{a+x}$ into a series.

Ans. $1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \frac{x^5}{a^5} + \&c.$

5. Develop $\sqrt{1+x}$ into a series.

$$Ans.\ \pm\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\&c.\right).$$

6. Develop $\sqrt{x+1}$ into a series.

$$Ans.\ \pm\left(\sqrt{x}+\frac{1}{2\sqrt{x}}-\frac{1}{8\sqrt{x^3}}+\frac{1}{16\sqrt{x^5}}-\&c.\right).$$

7. Develop $\sqrt{2}=\sqrt{1+1}$ into a series.

$$Ans.\ \pm(1+\tfrac{1}{2}-\tfrac{1}{8}+\tfrac{1}{16}-\tfrac{5}{128}+\&c.).$$

8. Develop $\sqrt{5}=\sqrt{4+1}$ into a series.

$$Ans.\ \pm(2+\tfrac{1}{4}-\tfrac{1}{64}+\tfrac{1}{512}-\&c.).$$

9. Develop $(m^4+n^4)^{\frac{1}{4}}$ into a series.

$$Ans.\ \pm(m+\tfrac{1}{4}m^{-3}n^4-\tfrac{3}{32}m^{-7}n^8+\tfrac{7}{128}m^{-11}n^{12}-\&c.).$$

10. Develop $(a+x)^{\frac{1}{4}}$ into a series.

$$Ans.\ \pm(\sqrt[4]{a}+\tfrac{1}{4}\frac{x}{\sqrt[4]{a^3}}-\tfrac{3}{32}\frac{x^2}{\sqrt[4]{a^7}}+\tfrac{7}{128}\frac{x^3}{\sqrt[4]{a^{11}}}.$$

11. Develop $(m^3+x^3)^{\frac{1}{3}}$ into a series.

$$Ans\ \ m\left(1+\frac{x^3}{3m^3}-\frac{x^6}{9m^6}+\frac{5x^9}{81m^9}-\&c.\right).$$

12. Develop $(1+x^3)^{\frac{1}{3}}$ into a series.

$$Ans.\ 1+\frac{x^3}{3}-\frac{x^6}{9}+\frac{5x^9}{81}-\&c.$$

13. Develop $\dfrac{m}{\sqrt{n^2+x^2}}$ into a series.

$$Ans.\ \pm\frac{m}{n}\left(1-\frac{x^2}{2n^2}+\frac{3x^4}{8n^4}-\frac{5x^6}{16n^6}+\frac{35x^8}{128n^8}-\&c.\right).$$

14. Develop $\sqrt[5]{a+x}$ into a series.

$$Ans.\ \sqrt[5]{a}+\frac{1}{5}\cdot\frac{x}{\sqrt[5]{a^4}}-\frac{2}{25}\cdot\frac{x^2}{\sqrt[5]{a^9}}+\frac{6}{125}\cdot\frac{x^3}{\sqrt[5]{a^{14}}}-\frac{21}{625}\cdot\frac{x^4}{\sqrt[5]{a^{19}}}+\&c.$$

15. Develop $\sqrt[5]{1+x}$ into a series.

$$Ans.\ 1+\frac{1x}{5}-\frac{2x^2}{25}+\frac{6x^3}{125}-\frac{21x^4}{625}+\&c.$$

16. Develop $\sqrt[5]{1-x}$ into a series.

$$Ans.\ 1 + \frac{1x}{5} + \frac{2x^2}{25} + \frac{6x^3}{125} + \frac{21x^4}{625} + \&c.$$

17. Develop $\sqrt[5]{2}$ into a series.

$$Ans.\ 1 + \tfrac{1}{5} - \tfrac{2}{25} + \tfrac{6}{125} - \tfrac{21}{625} + \&c.$$

CONSEQUENCES OF THE BINOMIAL FORMULA—SQUARE OF ANY POLYNOMIAL.

381. We have seen that $(a+b)^2 = a^2 + 2ab + b^2$, that is, the square of a binomial, is equal to the sum of the squares of its two terms, plus the double product of those terms. To find the square of a trinomial; $a + b + c$, let $a + b = s$. Then, $(a + b + c)^2 = (s + c)^2 = s^2 + 2sc + c^2 = (a + b)^2 + 2c(a + b) + c^2 = a^2 + b^2 + c^2 + 2ca + 2cb + 2ab$; that is, the square of a trinomial is equal to the sum of the squares of its three terms, plus the double product of these terms taken two and two. To find the square of a polynomial of four terms, $a + b + c + d$, let $a + b + c = s'$ Then, $(a + b + c + d)^2 = (s' + d)^2 = s'^2 + 2s'd + d^2 = a^2 + b^2 + c^2 + d^2 + 2da + 2db + 2dc + 2ca + 2cb + 2ab$; that is, equal to the sum of the squares of all its terms, plus the double product of these terms taken two and two. Now, it is evident that the same law will hold good for the square of a polynomial composed of five terms. For this square is composed of the square of the first four terms plus the square of the fifth term, plus the double product of the fifth term by the sum of the first four terms. And, since the square of the first four terms will give the sum of the squares of all these terms, plus the double product of these terms, taken two and two, it is evident that, when the double product of the fifth term by the sum of the other four, is added to the other double products, we will have the double product of all the terms, taken two and two; and it is plain, moreover, that when the square of the fifth term is added to the sum of the squares of the other four terms, that we will have the sum of the squares of all the terms. The law is then true for five terms, and it is plain that the same reasoning can be extended to six, seven, and any number of terms. Hence, we conclude, *that the square of any polynomial is equal to the sum of the squares of all the terms, plus the double product of all the terms, taken two and two.*

EXAMPLES.

1. Required the square of $a + b + c + d + e$.

Ans. $a^2 + b^2 + c^2 + d^2 + e^2 + 2ab + 2ac + 2ad + 2ae + 2bc + 2bd + 2be + 2cd + 2ce + 2de$.

2. Required the square of $a + b + c + d + e + f$.

Ans. $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 2ab + 2ac + 2ad + 2ae + 2af + 2bc + 2bd + 2be + 2bf + 2cd + 2ce + 2cf + 2de + 2df + 2ef$.

CUBE OF ANY POLYNOMIAL.

382. The cube of $(a + b)$, is $a^3 + 3a^2b + 3ab^2 + b^3$.

To find the cube of $a + b + c$, let $a + b = s$. Then, $(a + b + c)^3 = (s + c)^3 = s^3 + 3s^2c + 3sc^2 + c^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 3c^2a + 3c^2b + 6abc$; that is, the cube of a trinomial is composed of the sum of the cubes of its three terms, plus the sum of the products arising from multiplying three times the square of each term into the first power of the other terms, plus six times the product of all the terms, taken three and three.

By pursuing precisely the same course of reasoning that we have already employed, it can easily be shown that the law is general for any polynomial.

EXAMPLES.

1. Required the cube of $a + b + c + d$.

Ans. $a^3 + b^3 + c^3 + d^3 + 3a^2b + 3a^2c + 3a^2d + 3b^2a + 3b^2c + 3b^2d + 3c^2a + 3c^2b + 3c^2d + 3d^2a + 3d^2b + 3d^2c + 6abc + 6abd + 6acd + 6bcd$.

2. Required the cube of $a + b + c + d + e$.

Ans. $a^3 + b^3 + c^3 + d^3 + e^3 + 3a^2b + 3a^2c + 3a^2d + 3a^2e + 3b^2a + 3b^2c + 3b^2d + 3b^2e + 3c^2a + 3c^2b + 3c^2d + 3c^2e + 3d^2a + 3d^2b + 3d^2c + 3d^2e + 3e^2a + 3e^2b + 3e^2c + 3e^2d + 6abc + 6abd + 6abe + 6bcd + 6bce + 6cde + 6acd + 6ace + 6ade + 6bde$.

EXTRACTION OF THE *Nth* ROOT OF WHOLE NUMBERS AND POLYNOMIALS.

383. The most important consequence of the binomial formula is, that we are enabled by it to extract high roots of whole numbers and polynomials. We will begin with the simplest case, the extraction of the n^{th} root of whole numbers.

EXTRACTION OF THE *Nth* ROOT OF WHOLE NUMBERS.

Let a represent the tens, and b the units of the required root. Then, the given number will be $(a + b)^n$, and from the binomial formula we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{1 \cdot 2} + \frac{n(n-1)(n-2)a^{n-3}b^3}{1 \cdot 2 \cdot 3} + \&c.$$

That is, a number, whose n^{th} root is composed of tens and units, is made up of the n^{th} power of the tens, plus n times the $(n-1)^{th}$ power of the tens into the first power of the units, plus, &c.

It is evident that the n^{th} power of the tens will give, at least, $n + 1$ figures, therefore, the n^{th} root of the tens cannot be sought in the n right hand figures. These must, therefore, be cut off from the right. We next seek the greatest n^{th} root contained in the left hand period, and, when we have found it, we will have a of the formula. Raise the root found to the n^{th} power, and subtract this power from the left hand period. The remainder will correspond to $na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{1 \cdot 2} +$ &c., of the formula. The approximate divisor to find b (the unit of the root) is na^{n-1}. The true divisor is $na^{n-1} + \frac{n(n-1)a^{n-2}b}{1 \cdot 2.} + \frac{n(n-1)(n-2)a^{n-3}b^2}{1 \cdot 2 \cdot 3} +$ &c; but, as b is unknown, we can only use the approximate divisor. The $n-1$ figures, on the right of the remainder, must be separated from the other figures, because n times the $(n-1)^{th}$ power of the tens will give, at least, n figures. Dividing the period on the left by the approximate divisor, we get the units of the root, or, generally, a number greater than the units, because our divisor is too small. It is plain that the number of terms rejected when using the approximate divisor, depends upon the number of units in n, and that the value of each term depends mainly upon a. When, therefore, n and a are both large, or when one only is large, the approximate divisor is very considerably too small, and the quotient, therefore, will be too great. Raise the two figures of the root found to the n^{th} power, and compare the result with the given number; if it be greater than this number, the last figure of the root must be reduced by one, or more. Let us illustrate by an example.

Required the fifth root of 33554432.

$$\begin{array}{r|l} 335\ 54432 & 32 \\ 243 & \\ \hline na^{n-1} = 5 \times 81 = 405 \quad 92\dot{}54432 & \end{array}$$

We began by cutting off the five right hand figures. The next step was to find 3, the greatest fifth root contained in 335, the left hand period. Next, the approximate divisor, 405, was formed, and this was found to enter twice in the left hand period of the remainder, after subtracting the n^{th} power of the tens from the given number. Upon trial, 32 is found to be the true root, for, $(32)^5 = 33554432$. In this case, the unit figure of the root not being large in comparison with the tens, the approximate did not differ so materially from the true as to give a quotient figure too great by unity, or some other number. But, suppose it were required to extract the fourth root of 230625.

Then,

$$\begin{array}{r|l} 39\dot{}0625 & 25 \\ 16 & \\ \hline na^{n-1} = 4 \times (2)^3 = 32,\ 230\dot{}625 & \end{array}$$

The approximate divisor gives 7 as a quotient; but, upon trial, it has to be diminished by 2; and 25, not 27, is the true root.

It will be seen that the units of the root can only be ascertained by trial. If the units of the root be large in comparison with the tens, the quotient obtained by dividing the left hand period of the remainder by the approximate divisor, will often differ considerably from the true units of the root. The following example will illustrate this:

$$\sqrt[4]{13\dot{}0321} = 19$$

$$\begin{array}{r} 1 \\ \hline \end{array}$$

$$na^{n-1} = 4\ .\ 1^3 = 4 \qquad 120\dot{}321 = na^{n-1}b + \frac{n\,(n-1)a^{n-2}b^2}{1\ .\ 2} + \&c.$$

In this example, the approximate divisor gives 30 as a quotient, but this is absurd. We try 9, the greatest figure of the units, and find it to be right; for $(19)^4 = 130321$. The quotient, 30, being so large, we concluded that the units of the root must be large, and, therefore, tried 9.

Our reasoning has been confined to numbers whose root contained but two figures; but, as in the corresponding cases of the extraction of the square root and cube root of whole numbers, it can readily be extended to numbers whose root contains any number of figures. It is not necessary to repeat the generalization of the principles, and we, therefore, pass at once to the rule for the extraction of the n^{th} root of whole numbers.

RULE.

I. *Divide the given number into periods of* n *figures each, beginning on the right. Extract the* n^{th} *root of the greatest* n^{th} *power contained in the left hand period, and set this root on the right, after the manner of a quotient in division. Subtract the* n^{th} *power of the root so found from the left hand period.*

II. *To this remainder annex the next period, and separate from the new number so formed the* $n - 1$ *figures on the right. Regard the left hand period as a dividend.*

III. *Divide the dividend by* n *times the* $(n -)^{th}$ *power of the first figure of the root. The quotient will be the second figure of the root, or a number greater than the true second figure of the root. Raise the two figures of the root found to the* n^{th} *power. If the result exceed the two left hand periods of the given number, the last figure of the root must be reduced until the* n^{th} *power of the two figures is equal to, or something less than the two left hand periods.*

IV. *Annex the third period to the remainder, after subtracting the* n^{th} *power of the first two figures of the root from the two left hand periods, and cut off* $n - 1$ *figures from the right of the new number thus formed, and regard the left hand period as a new dividend.*

V. *Divide this dividend by* n *times the* $(n - 1)^{th}$ *power of the first two figures of the root, the quotient will be the third figure of the root, or a number greater than the third figure. Ascertain, by trial, whether the last figure is correct, and proceed in this way until all the periods are brought down. The number of figures in the root will always be equal to the number of periods in the given number.*

EXAMPLES.

1. Required the $\sqrt[4]{68574961}$. *Ans.* 91.

2. Required the $\sqrt[5]{6240321451}$. *Ans.* 91.

3. Required the $\sqrt[5]{2476099}$. *Ans.* 19.

4. Required the $\sqrt[4]{108243216}$. *Ans.* 102.

5. Required the $\sqrt[5]{1692662195786551}$. *Ans.* 1111.

6. Required the $\sqrt[5]{539218609632}$. *Ans.* 222.

7. Required the $\sqrt[7]{35831808}$. *Ans.* 12.

8. Required the $\sqrt[7]{587068342272}$. *Ans.* 48.

9. Required the $\sqrt[7]{75154747810816}$. *Ans.* 96.

10. Required the $\sqrt[5]{541651902651696 32}$. *Ans.* 2222.

APPROXIMATE ROOT OF AN IRRATIONAL NUMBER TO WITHIN A CERTAIN VULGAR FRACTION.

384. The principles of approximation have been so fully explained under the head of the Square Root and Cube Root, that it will only be necessary now to give the rule for the approximate n^{th} root to within a vulgar fraction, whose numerator is unity.

RULE.

Multiply and divide the given number by the n[th] *power of the denominator of the fraction that marks the degree of approximation. The root of the numerator of the new fraction thus formed, to within the nearest unit, divided by the exact root of the denominator, will be the approximate root required.*

EXAMPLES.

1. Required $\sqrt[4]{6}$ to within $\frac{1}{2}$. *Ans.* $\frac{3}{2}$.

For, $\sqrt[5]{4} = \sqrt[4]{6 \times \frac{16}{16}} = \sqrt[4]{\frac{96}{16}} = \frac{3}{2}$.

2. Required $\sqrt[5]{9}$ to within $\frac{1}{2}$. *Ans.* $\frac{3}{2}$.

3. Required $\sqrt[7]{90}$ to within $\frac{1}{4}$. *Ans.* $\frac{7}{4}$.

4. Required $\sqrt[7]{2098152}$ to within $\frac{1}{5}$. *Ans.* $\frac{40}{5}$, or 8, nearly.

5. Required $\sqrt[11]{8589929092}$ to within $\frac{1}{3}$. *Ans.* $\frac{24}{3}$, or 8, nearly.

APPROXIMATE ROOT OF WHOLE NUMBERS TO WITHIN A CERTAIN DECIMAL.

RULE.

385. *Annex as many periods of ciphers (each period consisting of* n *figures,) as there are places required in the root. Extract the* n^{th} *root of the new number thus formed to within the nearest unit, and point from the right for decimals, the number of places required in the root.*

EXAMPLES.

1. Required $\sqrt[5]{39}$ to within .1. *Ans.* 2.1

2. Required $\sqrt[5]{116857201}$ to within .1. *Ans.* 41.1.

3. Required $\sqrt[7]{9998989}$ to within .01. *Ans.* 9.99.

4. Required $\sqrt[5]{258991}$ to within .1. *Ans.* 12.1.

5. Required $\sqrt[9]{511.99999999999999888}$ to within .01. *Ans.* 1.99.

APPROXIMATE *Nth* ROOT OF A MIXED NUMBER TO WITHIN A CERTAIN DECIMAL.

RULE.

386. *Annex ciphers, if necessary, until the decimal part will contain as many periods of* n *figures each as there are places required in the root. Extract the* n^{th} *root, and point from the right the required number of decimal places.*

EXAMPLES.

1. Required $\sqrt[5]{1.6051}$ to within .1. *Ans.* 1.1, nearly.

2. Required $\sqrt[5]{2.4884}$ to within .1. *Ans.* 1.2.

3. Required $\sqrt[7]{3.583181}$ to within .1. *Ans.* 1.2.

4. Required $\sqrt[5]{62403.21461}$ to within .1. *Ans.* 9.1.

5. Required $\sqrt[5]{392.5430946993}$ to within .01. *Ans.* 3.31.

6. Required $\sqrt[5]{1725.4995508234}$ to within .01. *Ans.* 4.44.

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APPROXIMATE *Nth* ROOT OF NUMBERS ENTIRELY DECIMAL.

RULE.

387. *Annex ciphers, if necessary, until there are as many periods of* n *figures each as there are places required in the root. Extract the* n[th] *root of the new number thus formed, and point off the required number of decimal places.*

EXAMPLES.

1. Required $\sqrt[5]{.00245}$ to within .1. *Ans.* .3.

2. Required $\sqrt[5]{.0028629161}$ to within .01. *Ans.* .31.

3. Required $\sqrt[5]{.0004084201}$ to within .01. *Ans.* .21.

4. Required $\sqrt[7]{.51676701935}$ to within .01. *Ans.* .91.

5. Required $\sqrt[4]{.96059690}$ to within .01. *Ans.* .99.

6. Required $\sqrt[5]{.1245732577}$ to within .01. *Ans.* .66.

7. Required $\sqrt[6]{.464404086785}$ to within .01. *Ans.* .88.

8. Required $\sqrt[11]{.0000000000285311670612}$ to within .01. *Ans.* .11.

9. Required $\sqrt[11]{.00048828126}$ to within .1. *Ans.* .5.

10. Required $\sqrt[11]{.000000000000000000000285311670612}$ to within .001. *Ans.* .011.

11. Required $\sqrt[5]{.2706784158}$ to within .01. *Ans.* .77.

PERMUTATIONS AND COMBINATIONS.

388. Permutations are the results obtained by writing *n* letters in sets of 1, 2, 3, ... or *n* letters, so that each set shall differ from all the other sets in the order in which the letters are taken. Thus, the permutations of the three letters, *a*, *b*, and *c*, taken singly, are *a*, *b*, *c*; and in sets of two letters each, *ab*, *ac*, *ba*, *bc*, *ca*, *cb*; and in sets of three letters each, *abc*, *acb*, *bac*, *bca*, *cba*, *cab*.

It will be seen that the sets differ only in the manner in which the letters are written, the letters in all the sets being the same.

389. *Let it be required to determine the number of permutations of* m *letters, taken* n *in a set.*

Suppose the letters to be a, b, c, d, &c., and let us first permute them singly, we will evidently have m permutations of the m letters taken one in a set. Now, let us reserve a, there will remain $m - 1$, letters b, c, d, &c., which, permuted singly, will give $(m - 1)$ permutations, b, c, d, e, f, &c.: write a before each of these letters, and we will have $m - 1$ permutations, ab, ac, ad, &c., of two letters, with a as the first letter of each set. Next, let us reserve b out of the letters, a, b, c, d, &c., and then permute a, c, d, &c., singly. We will again have $(m - 1)$ permutations of the letters, taken in sets of one letter each. Writing b before each of these sets, we will have $(m - 1)$ permutations of m letters, taken in sets of two, with b as the first letter. Reserving c in like manner, we can again form $(m - 1)$ permutations of m letters, taken two in a set, with c as the first letter of each set. Reserving all the m letters in succession, we can evidently form $m(m - 1)$ permutations of m letters, taken 2 in a set. And, of these, a will be the first letter of $(m - 1)$ sets, b the first letter of $(m - 1)$ sets, c the first letter of $(m - 1)$ sets, and so on. It is plain, moreover, that a will not only be the first letter of $(m - 1)$ sets, but that it will also be the last letter of $(m - 1)$ sets, and that it therefore has been made to occupy all possible positions. As the same remark may be made of b and all the other letters, it is obvious that $m(m - 1)$ truly expresses the number of permutations that can be made of m letters, taken two in a set. *That is, the number of permutations of* m *letters, taken two in a set, is equal to the total number of letters into the total number of letters, less one.*

In like manner, we may find the number of permutations of m letters, taken three in a set. For, if we omit one of the letters, as a, there will be $m - 1$ letters left, and these, permuted in sets of two letters, will give $(m-1)(m - 2)$ permutations. For, we have just seen that the number of permutations, taken in sets of two, was expressed by the number of letters into the number of letters, less one. Writing the reserved letter, a, before each of these $(m - 1)(m - 2)$ sets, we will form $(m - 1)(m - 2)$ permutations of m letters, taken three in a set, with a as the first letter of each set. Reserving in succession each of the m letters, we can plainly form $m(m-1)(m-2)$ permutations of m letters, taken three in a set.

By the same course of reasoning it can be readily shown that the number of permutations of m letters, taken four in a set, will be expressed by $m(m-1)(m-2)(m-3)$; and of m letters, taken five in a set, by $m(m-1)(m-2)(m-3)(m-4)$.

We observe, in all these expressions, that the first factor is the number of letters, and that the last factor is the number of letters, diminished by the number in a set less one. Moreover, each factor after the first is one less than the preceding factor. Hence, the number of permutations of m letters, taken n together, will be expressed by $m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)$, or, denoting by A the number of permutations of m letters, taken n in a set, we have $A = m(m-1)(m-2)\ldots\ldots(m-n+1)$.

To apply this formula, it is best to determine in the first place the *last factor*, we then know where to stop. Since the first factor is always the number of letters, and the law respecting the mean factors is known, the formula can then be readily applied.

EXAMPLES

1. Required the number of permutations of 10 letters, taken 7 and 7.
Ans. $10(10-1)(10-2)(10-3)(10-4)(10-5)(10-6)$, or $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 604800$.

For, $n = 7$, and $m = 10$, then $m - n + 1 = 10 - 7 + 1 = 10 - 6$, the last factor is then 4, and the first 10, the intermediate factors can easily be formed.

2. Required the number of permutations of 5 letters, taken 4 and 4.
Ans. $5(5-1)(5-2)(5-3) = 5 \times 4 \times 3 \times 2 = 120$.

3. Required the number of permutations of 12 letters, taken 10 and 10.
Ans. $12(12-1)(12-2)(12-3)(12-4)(12-5)(12-6)(12-7)(12-8)(12-9)$.

4. Required the number of permutations of 12 letters, taken 11 and 11. *Ans.* $12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$.

5. Required the number of permutations of 12 letters, taken 12 in a set.
Ans. $12(12-1)(12-2)(12-3)(12-4)(12-5)(12-6)(12-7)(12-8)(12-9)(12-10)(12-11)$; or, reversing the factors, $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12$.

Remarks.

390. If, in the general formula, $A = m(m-1)(m-2) \dots (m-n+1)$, we make $m = n$, and call B the corresponding value of A, we will have $B = n(n-1)(n-2) \dots 4 \times 3 \times 2 \times 1$; or, reversing the order of the factors, $B = 1 \times 2 \times 3 \times 4 \dots \dots (n-2)(n-1)n$. Hence, the number of permutations of n letters, taken all together, is equal to the product of all the natural numbers from 1 to n, inclusive.

Example 5 is an illustration of this.

It is plain that, by increasing n (m remaining constant), we will increase the number of permutations, because we will have more factors in the product that expresses the number of permutations. And, when n is made equal to m, this product is the greatest possible. Thus, the result is greater in example 4 than in 3, and greater in 5 than in 4.

When $n = m + 1$, the whole formula reduces to zero. This plainly *ought to be so*, since it is impossible to permute m letters in sets of $m + 1$ letters. Zero, here, as in many other places, is the symbol of impossibility.

When $n = 1$, the last factor, $m - n + 1$, is equal to m. Hence, the first and last factors are the same, and we have $A = \frac{m}{1} = m$; that is, the number of permutations of m letters, taken one in a set, is equal to m.

COMBINATIONS.

391. Combinations are the results obtained by writing any number of letters, as m, in sets of 1 and 1, 2 and 2, 3 and 3, ... n and n, m and m, so that each set shall differ from all the other sets by at least one letter. In *permutations*, the sets differ in the *order* in which the letters are written; in *combinations*, the sets differ in the *letters themselves*. Thus, the letters, *abc*, taken all together, give but one combination, *abc*; but will give six permutations,

abc, acb, bac, bca, cab, cba.

If the same letters are taken two and two, they will give but three combinations

ab, ac, bc,

while each combination is susceptible of two permutations; and there are, therefore, six permutations of the three letters, taken two and two, thus,

$$ab \text{ will give } \left.\begin{matrix} ab, \\ ba, \end{matrix}\right\} \quad ac \text{ will give } \left.\begin{matrix} ac, \\ ca, \end{matrix}\right\} \quad \text{and } bc \text{ will give } \left.\begin{matrix} bc, \\ cb. \end{matrix}\right\}$$

392. *Let it be required to determine the total number of combinations of* m *letters, taken* n *in a set.*

It is evident, from what has been shown, that if we knew the total number of combinations of m letters, taken n in a set, that we could get the number of permutations of m letters taken n in a set, by multiplying the number of combinations by the number of permutations obtained by permuting *each* combination; or, in other words, by multiplying the number of combinations of m letters, taken n in a set, by the number of permutations of n letters, taken all together. Conversely, when the number of permutations of m letters, taken n in a set, and the number of permutations of n letters, taken all together, are known, we can, by dividing the former by the latter, determine the number of combinations of m letters, taken n in a set.

393. To illustrate more fully by an example.

Let us combine the four letters, a, b, c and d, in sets of three, we will have the four combinations,

$$abc, \; abd, \; acd, \; bcd.$$

If, now, we permute each of these sets, taking all the letters in each set, we will have the total number of permutations of four letters, taken three in a set, that is, the total number of permutations of m letters, taken n in a set.

Writing the results in tabular form, we have

Combinations of a, b, c, d, in sets of 3.........	*abc*	*abd*	*acd*	*bcd*
Permutations of each combination, taken 3 and 3, or all together.	*abc*	*abd*	*acd*	*bcd*
	acb	*bad*	*adc*	*bdc*
	bac	*bda*	*cad*	*cbd*
	bca	*dab*	*cda*	*cdb*
	cab	*dba*	*dac*	*dbc*
	cba	*adb*	*dca*	*dcb*

We see that each of the four combinations, in sets of 3, gives six permutations, taken 3 and 3. There will, therefore, be $4 \times 6 = 24$ permutations of four letters, taken three in a set. Now, as a corresponding table could be formed for any number of letters, it is plain that the total number of permutations of m letters, taken n in a set, is equal to the number of combinations of m letters, taken n in a set, multiplied by the number of permutations of n letters, taken all together.

Hence, if

$X =$ the number of combinations of m letters, taken n in a set;

$Y =$ the number of permutations of n letters, taken all together;

$Z =$ the total number of permutations of m letters, taken n in a set,

we shall have $Z = X \,.\, Y$. Hence, $X = \frac{Z}{Y}$.

But Z and Y are already known.

For $Z = m(m-1)(m-2) \ldots\ldots (m-n+1)$, Formula (A); and $Y = 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \ldots\ldots n$, Formula (B). Hence, we have $X = \frac{m(m-1)\ldots\ldots(m-n+1)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots n}$. From which, we conclude that the number of combinations of m letters, taken n in a set, is equal to the number of permutations of m letters, taken n in a set, divided by the number of permutations of n letters, taken n in a set.

In the application of this formula, it will be of service to remember that the first factor of the numerator is the number of letters, that each successive factor is one less than the preceding, and that the last factor is the number of letters diminished by the number in a set, less one. It is well, then, to determine first the extreme factors of the numerator; the mean factors can then be readily supplied.

In regard to the denominator, the last factor is the number in a set, and the factors counted to the left go on diminishing by unity unto the first factor, which is always unity.

EXAMPLES.

1. Required the number of combinations of 6 letters, taken 4 in a set.

$$Ans.\ \frac{6(6-1)(6-2)(6-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} = \frac{6 \,.\times 5 \times 4 \times 3}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 15.$$

For $m = 6$ and $n = 4$. Hence, the first factor of the numerator is 6, and the last $m - n + 1 = 6 - 4 + 1 = 6 - 3$. The two mean factors of the numerator are then readily formed, the extremes being known.

2. Required the number of combinations of 12 letters, taken 7 and 7.

$$Ans.\ \frac{12(12-1)(12-2)(12-3)(12-4)(12-5)(12-6)}{1.2.3.4.5.6.7}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 11$$

$\times\ 9 \times 8 = 792.$

3. Required the number of combinations of 12 letters, taken 5 and 5.

$$Ans.\ \frac{12(12-1)(12-2)(12-3)(12-4)}{1 \times 2 \times 3 \times 4 \times 5} = \frac{12 \times 11 \times 10 \times 9 \times 8}{12 \times 10}$$

$= 792.$

4. Required the number of combinations of 12 letters, taken 9 and 9.

$$Ans.\ \frac{12(12-1)(12-2)(12-3)(12-4)(12-5)(12-6)(12-7)(12-8)}{1.2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9} =$$

$$\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9} = \frac{1320}{6} = 220.$$

5. Required the number of combinations of 12 letters, taken 3 and 3

$$Ans.\ \frac{12(12-1)(12-2)}{1.2.3} = \frac{12 \times 11 \times 10}{6} = 220.$$

6. Required the number of combinations of 10 letters, taken 9 and 9.
Ans. 10.

7. Required the number of combinations of 10 letters, taken 1 and 1.
Ans. 10.

8. Required the number of combinations of 25 letters, taken 21 and 21. *Ans.* 12650.

9. Required the number of combinations of 25 letters, taken 4 and 4.
Ans. 12650.

10. Required the number of combinations of 50 letters, taken 47 and 47. *Ans.* 19600.

11. Required the number of combinations of 50 letters, taken 3 and 3.
Ans. 19600.

Scholium.

394. The last ten examples show that the same number of letters, combined differently, may produce the same number of combinations, that is, the number of combinations of m letters, taken p in a set, may be equal to the number of combinations of m letters, taken q in a set. We propose to show that this will always be so when $m = p + q$. Thus,

in example 10, $m = 50$ and $p = 47$; in example 11, $m = 50$ and $q = 3$. The results have been the same in those examples, because $m = p + q$.

To show this, suppose $p > q$. Then the number of combinations of m letters, taken p in a set, will be expressed thus:

$$D = \frac{m(m-1) \ldots (m-q+1)(m-q)(m-q-1) \ldots\ldots (m-p+1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots (p-1)\, p}.$$

And for the number of combinations of m letters, taken q in a set, we will have the formula,

$$C = \frac{m(m-1)(m-2) \ldots\ldots (m-q+1)}{1 \,.\, 2 \,.\, 3\, (q-1)\, q}.$$

Hence, $D = C \frac{(m-q)(m-q-1) \ldots (m-p+1)}{(q+1)(q+2) \ldots p}$.

Now, it is plain that D will be equal to C, when $(m-q)(m-q-1)$ &c. $= p(p-1) \ldots (q+2)(q+1)$. Or, since the factors in both members go on decreasing by unity, and since the number of them is also equal, the last equation will be true when $m - q = p$, or $m = q + p$, as enunciated.

This rule is of importance whenever $p > \frac{m}{2}$.

For, then, we have only to take the difference between p and m, and the number of combinations of m letters, taken $m-p$ and $m-p$, will be the same as the number of combinations of m letters, taken p and p. Thus, in example 10, instead of taking the number of combinations of 50 letters, taken 47 in a set, we may take the number of combinations of 50 letters, taken 50—47, or 3 in a set.

In like manner, the number of combinations of 100 letters, taken 90 in a set, would be the same as the number of combinations of 100 letters, taken 10 in a set.

395. If, in the formula, $D = C \frac{(m-q)(m-q-1) \ldots (m-p+1)}{(q+1)(q+2) \ldots p}$, we make $q = p - 1$, we will have $D = C \frac{(m-p+1)}{p}$; for, in that case, D will have only one more factor than C. *Hence, the number of combinations of* m *letters, taken* p *and* p, *is equal to the number of combinations of* m *letters, taken* p — 1 *and* p — 1, *multiplied by the factor* $\frac{m-p+1}{p}$ When $p = 2, 3, 4$, &c., or the sets are made up of 2, 3, 4, &c., letters, the factor $\frac{m-p+1}{p}$ will become $\frac{m-1}{2}$,

$\frac{m-2}{3}, \frac{m-3}{4}$. Suppose, for example, that we have 8 letters, or $m = 8$, then the successive multipliers will be $\frac{7}{2}, \frac{6}{3}, \frac{5}{4}, \frac{4}{5}, \frac{3}{6}, \frac{2}{7}$. And, since the number of combinations of 8 letters, taken 1 and 1, is 8, we will have the number of combinations, taken 2 and 2, 3 and 3, 4 and 4, &c., expressed $8 \times \frac{7}{2} = 28$, $28 \times \frac{6}{3} = 56$, $56 \times \frac{5}{4} = 70$, $70 \times \frac{4}{5} = 56$, $56 \times \frac{3}{6} = 28$, $28 \times \frac{2}{7} = 8$.

This remarkable law has had many important applications, one of the principal of which is, the determining of the coefficients in the binomial formula.

It is plain from what has been shown that, if we write in succession the number of combinations of m letters, taken 1 and 1, 2 and 2, ... n and n, we will have a series of numbers increasing to the middle term, and then repeated in retrograde order.

Thus, 6 letters combined, 1 and 1, 2 and 2, 3 and 3, 4 and 4, 5 and 5, give the series 6, 15, 20, 15, 6. In like manner, 7 letters give the series, 7, 21, 35, 35, 21, 7. The law of formation is precisely that which we have observed in the binomial development; and, in fact, in this development, the coefficients, after the second, are the combinations of m letters, taken 2 and 2, 3 and 3, &c., ... n and n.

396. Let us now seek the greatest term of the series formed by combining m letters, 1 and 1, 2 and 2, ... n and n. The factors, which determine the successive terms, are $\frac{m-1}{2}, \frac{m-2}{3}, \ldots \frac{m-n+1}{n}$, &c. Now, since these numerators go on decreasing, and the denominators increasing, it is evident that the successive products will go on increasing until $m - n + 1 = n$, and after that will decrease and be repeated in reverse order. Suppose m an odd number, then placing $m - n + 1 = n$, we get $n = \frac{1}{2}(m + 1)$. Now, if we include unity, the last term of the series, there will be m terms in all, and in case of m being odd, the n^{th} term will obviously be even, and, of course, have an even number of terms preceding and succeeding it. Thus, in the series, 7, 21, 35, 35, 21, 7, 1; $n = \frac{m+1}{2} = 4$, will represent the 4th term, and we see that it has three terms before and three after it. It is plain, moreover, that the n^{th} term, being formed from the $(n-1)^{\text{th}}$ term by multiplying the latter by $\frac{m-n+1}{n} = 1$, is, also, equal to the $(n-1)^{\text{th}}$ term.

Hence, when m is an odd number, there will be two equal central terms, the series will go on increasing unto the first of these, and decreasing from the second unto the last term, unity. Thus, the combinations of 5 letters, taken 1 and 1, 2 and 2, 3 and 3, 4 and 4, 5 and 5, give the series 5, 10, 10, 5, 1.

397. When m is even, the factors increase until $n = \frac{1}{2}m$. We cannot, in this case, have $n = \frac{m+1}{2}$, for that would make n fractional, a manifest absurdity. No factor, in this case, is unity, and, of course, there are no equal central terms. The terms go on increasing until $n = \frac{1}{2}m$, and are then reproduced in reverse order. It is obvious that whenever $n > \frac{1}{2}m$, the factor $\frac{m-n+1}{n}$ will become a proper fraction, and, of course, the term of the series formed by multiplying the preceding term by this factor, will be less than the preceding. Ten letters, taken 1 and 1, 2 and 2, &c., give the series, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1.

398. If we add together the values of C and D, as found in Art. 394, and at the same time make $q = p - 1$, we will have $C + D = C + \frac{C(m-p+1)}{p} = \frac{C(m+1)}{p} = \frac{m+1}{1} \times C \times \frac{1}{p} = \left(\frac{m+1}{1}\right) \left(\frac{m}{2}\right)\left(\frac{m-1}{3}\right) \ldots\ldots \frac{(m-p+3)}{p-1}\frac{(m-p+2)}{p}$.

The second member of this equation plainly denotes the number of combinations of $m+1$ letters, taken p and p; whilst the first member denotes the sum of the combinations of m letters, taken p and p, and $p-1$ and $p-1$. This important relation enables us to get the number of combinations of $m+1$ letters from that of m letters. Thus, make $m = 4$, $p = 3$, and $p - 1 = 2$, then 4 letters, taken 2 and 2, give 6 combinations, and taken 3 and 3, give 4, and we find that 5, or $m+1$ letters, taken p, or 3 in a set, give a number of combinations equal to $4 + 6$, or 10.

399. Upon this principle, the following table (p. 384) has been constructed. The first vertical column is made up of ones; the second column contains the natural numbers from 1 to 20, and expresses the number of combinations of these numbers, taken 1 and 1; the other vertical columns express the combinations of the same numbers, taken 3 and 3, 4 and 4, 5 and 5, &c. These vertical columns result from the horizontal, which are constructed according to the principle demonstrated in Article 398, observing that every horizontal column must close with unity. Thus, the numbers in the second horizontal column are 1, 2

and 1, the numbers in the next column are, $1 + 2 = 3$, $2 + 1 = 3$ and 1, and express the number of combinations of three letters, taken 1 and 1, 2 and 2, and 3 and 3. Now, prefix unity, and we have the third horizontal row made up of 1, 3, 3 and 1, and the fourth row will be made up of 1, $3 + 1 = 4$, $3 + 3 = 6$, $3 + 1 = 4$ and 1, the last four numbers expressing the number of combinations of four letters, taken 1 and 1, 2 and 2, 3 and 3, and 4 and 4. All the other horizontal columns are formed in the same manner.

ARITHMETICAL TRIANGLE OF PASCAL.

1	1	1								
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1
1	11	55	165	330	462	462	330	165	55	11
1	12	66	220	495	792	924	792	495	220	66
1	13	78	286	715	1287	1716	1716	1287	715	286
1	14	91	364	1001	2002	3003	3432	3003	2002	1001
1	15	105	455	1365	3003	5005	6435	6435	5005	3003
1	16	120	560	1820	4368	8008	11440	12870	11440	8008
1	17	136	680	2380	6188	12376	19448	24310	24310	19448
1	18	153	816	3060	8568	18564	31824	43758	48620	43758
1	19	171	969	3876	11628	27132	50388	75582	92378	92378
1	20	190	1140	4845	15504	38760	77520	125970	167960	184756
0	1 and 1	2 and 2	3 and 3	4 and 4	5 and 5	6 and 6	7 and 7	8 and 8	9 and 9	10 and 10

400. We can now readily find the entire sum of the combinations formed by any number of letters, taken in every possible way. Let the numbers in the t^{th} row be expressed by $1, a\ b \ldots m, m \ldots b, a, 1$; t being supposed an odd number. Then their sum will be $2(a + b \ldots\ldots + m + 1)$. The numbers in the next row will be expressed by $1 + a$, $a + b \ldots + . + m$, $+ m + m \ldots b + a$, $a + 1$ and 1, and their sum will be $4(a + b \ldots + m + 1) + 1$. Hence, the sum of the numbers in the $(t + 1)^{th}$ column, will be double of the preceding, and one more. When t is an even number, the same law can be shown to be true But the sum of the numbers in the second horizontal column, is $2^2 - 1$ Hence, in the third column, it will be $2(2^2 - 1) + 1 = 2^3 - 1$, and in the fourth, $2(2^3 - 1) + 1 = 2^4 - 1$. And it is plain that, in the t^{th} column it will be $2^t - 1$. It will be seen that the ones in the first

vertical column are omitted. Hence, $2^t - 1$, expresses the sum of the combinations of t letters, taken 1 and 1, 2 and 2 . . . . t and t. Thus, to apply the formula, let it be required to determine the number of combinations of three letters, taken 1 and 1, 2 and 2, 3 and 3. Then $t = 3$, and $2^t - 1 = 2^3 - 1 = 8 - 1 = 7$. This agrees with the fact, for the three letters, *a*, *b*, *c*, give us the seven combinations, *a*, *b*, *c*; *ab*, *ac*, *bc*, and *abc*. So, likewise, the four letters, *a*, *b*, *c*, *d*, by the formula, give a number of combinations equal to $2^4 - 1 = 16 - 1 = 15$; and we, in fact, have *a*, *b*, *c*, *d*; *ab*, *ac*, *ad*, *bc*, *bd*, *cd*; *abc*, *abd*, *acd*, *bcd*; and *abcd*.

401. We will now deduce an analogous formula to the above, for the sum of the permutations of n letters, taken 1 and 1, 2 and 2, 3 and 3, n and n, when each letter is combined with itself, so as to be taken to the second, third, fourth, and . . . n^{th} powers.

PERMUTATIONS IN WHICH THE LETTERS ARE REPEATED.

Two letters, *a* and *b*, permuted in this way, give the four permutations, *aa*, *ab*, *ba*, *bb*. Hence, the number of permutations of two letters, taken 2 and 2, when each letter is associated with itself, is equal to 2^2. Three letters, *a*, *b*, *c*, give the nine permutations, *aa*, *ab*, *ac*, *ba*, *ca*, *bb*, *bc*, *cb*, *cc*. Hence, the number of permutations of three letters, taken 2 and 2, is equal to 3^2. Four letters, *a*, *b*, *c*, *d*, give, when taken 2 and 2, the sixteen permutations, *aa*, *ab*, *ac*, *ad*, *ba*, *ca*, *da*, *bb*, *bc*, *bd*, *cb*, *db*, *cc*, *cd*, *dc*, *dd*. Hence, the number of permutations of four letters, taken 2 and 2, is equal to 4^2. And, in general, it is plain that the number of permutations of n letters, taken 2 and 2, is equal to n^2. We will next show that the number of permutations of n letters, taken 3 and 3, is equal to n^3. The three letters, *a*, *b*, and *c*, when permuted in this way, give the twenty-seven permutations,

aaa, *aab*, *aac*, *aba*, *baa*, *aca*, *caa*, *abc*, *cba*;
bbb, *bbc*, *bba*, *bcb*, *cbb*, *bab*, *abb*, *bac*, *bca*;
ccc, *ccb*, *cca*, *cbc*, *bcc*, *cac*, *acc*, *cab*, *acb*.

Hence, the number of permutations of three letters, taken 3 and 3, is equal to 3^3. And, in general, of n letters, taken 3 and 3, the number is n^3. In like manner, the number of permutations of n letters, taken 4 and 4, is expressed by n^4. Hence, for the entire sum of the permutations, taken 1 and 1, 2 and 2, 3 and 3 . . . . n and n, we have

$n + n^2 + n^3 \ldots\ldots + n^n = \frac{(n^n - 1)n}{n - 1}$, the whole constituting a geometrical series, whose common ratio is n. Thus, to apply the formula, suppose it be required to determine the number of permutations of five letters, then $n = 5$, and $\frac{(n^n - 1)n}{n - 1} = \frac{(5^5 - 1)5}{4} = \frac{15620}{4} = 3905$, and if $n = 24$, the number of letters in the alphabet, we have $\frac{(n^n - 1)n}{n - 1} = \frac{(24^{24} - 1)24}{23} = 1391724288887252999425128493402200$.

Thus, let it be required to determine the entire number of changes that can be made with the three vowels, *a*, *e*, and *i*. The formula $\frac{(n^n - 1)n}{n - 1}$, becomes $\frac{(3^3 - 1)3}{2} = \frac{26}{2} \cdot 3 = 39$, and we find that we have the thirty-nine permutations.

a, *e*, *i*, taken 1 and 1; *ae*, *ai*, *ea*, *ia*, *ei*, *ie*, *aa*, *ee*, *ii*, taken 2 and 2; *aei*, *eai*, *iea*, *eia*, *iae*, *aie*, taken 3 and 3, in the usual way; *aaa*, *aae*, *aai*, *eaa*, *iaa*, *aea*, *aia*, *eee*, *eea*, *eei*, *aee*, *iee*, *eae*, *eie*, *iii*, *iia*, *iie*, *aii*, *eii*, *iai*, *iei*, taken 3 and 3, when the letters are combined with themselves.

PARTIAL PERMUTATION.

402. Sometimes the nature of the problem is such, that some of the quantities cannot be permuted with each other. Thus, let it be required to determine how many words of two letters each can be formed out of the letters *a*, *e*, *c*, *d*, admitting that the consonants associated together will not form a word. The formula for the number of permutations of m letters, taken n and n, gives us $4(4 - 1) = 12$. But we have to reject the number of permutations of two letters, *c* and *d*, taken 2 and 2, or $2(2 - 1) = 2$. Hence, $12 - 2 = 10$, the number of permutations that can be formed in the required manner. They are, *ae*, *ac*, *ad*, *ea*, *ca*, *da*, *ec*, *ed*, *ce*, *de*.

RULE.

Find the entire number of permutations of n *letters, taken* n *and* n, *and from this subtract the number of permutations of the* p *especial letters, taken* n *and* n.

EXAMPLES.

1. How many words of two letters each can be formed out of the 4 first letters of the alphabet, assuming that consonants alone will not form a word?

Ans. $4(4-1)-3(3-1)=6$; words, *ab, ac, ad, ba, ca, da.*

2. How many numbers composed of 3 digits each can be formed from the number 24371, in such a way that none of the numbers contains only odd digits?

Ans. $5(5-1)(5-2)-3(3-1)(3-2)=54$. Numbers, 243, 234, 342, 324, 432, 423; 247, 274, 472, 427, 742, 724; 241, 214, 412, 421, 142, 124; 237, 273, 372, 327, 732, 723; 231, 213, 312, 321, 132, 123; 271, 217, 721, 712, 172, 127; 437, 473, 374, 347, 734, 743; 431, 413, 341, 314, 134, 143; 471, 417, 741, 714, 174, 147.

3. How many words of 6 letters each can be formed out of 6 vowels and 6 consonants, assuming that the consonants by themselves cannot form a word?

Ans. $12(12-1)(12-2)(12-3)(12-4)(12-5)-6(6-1)(6-2)(6-3)(6-4)(6-5)=664560$.

4. How many words of six letters each can be formed from the 20 consonants and 6 vowels of the alphabet, assuming that the consonants alone cannot form a word? *Ans.* 137858400.

GENERAL EXAMPLES IN PERMUTATIONS AND COMBINATIONS.

1. Three travellers on a journey reach three roads: in how many different positions, with respect to each other, may they continue travelling, provided, that no two of them pursue the same road?

Ans. 6. A may take the first road, B the second, and C the third; or, A the first, C the second, B the third, &c.

2. How many days can 3 persons be placed in a different position at dinner? *Ans.* 6 days.

3. How many days can 8 persons be placed in a different position at dinner? *Ans.* 40320 days.

4. In how many different positions can a stage driver place the 4 horses of his team in harness? *Ans.* 24.

5. How many changes may be made in the words of the sentence: "Lazy boys make worthless, vicious men?" *Ans.* 720.

6. How many changes may be struck with 10 keys of a piano? *Ans.* 3628800.

7. How many words of five letters each can be made out of the first 24 letters of the alphabet, assuming that no letter is repeated more than once in the same word? *Ans.* 5100480.

8. How many words of three letters each can be made out of the 24 first letters of the alphabet, with the same proviso as the preceding? *Ans.* 12144.

9. How many numbers of two digits each can be made out of the numbers 1243 (provided, that none of the digits of 1243 appear twice in the same number), and what are they?

Ans. 12. Numbers, 12, 43, 14, 13, 21, 34, 41, 31, 24, 23, 42, 32.

10. How many numbers of five digits each can be formed out of the number 1876543, provided, that no digit of the given number appear twice in the required results? *Ans.* 2520.

11. How many changes can be made in every file of 2 men in a squad of 20 men, so that the files shall differ by at least one man? *Ans.* 190.

12. How many changes can be made in the position of the first 12 letters of the alphabet? *Ans.* 479001600.

13. How many numbers of 1, 2, and 3 digits can be formed out of the number 476, provided, that no digit is repeated in the same number?

Ans. 15. Numbers, 4, 7, 6: 47, 46, 74, 64, 67, 76; 476, 467, 746, 764, 647, 674

14. How many numbers of 1, 2, and 3 digits can be formed out of the number 476, when each digit is repeated 1, 2, and 3 times in the respective numbers?

Ans. 39. Numbers, 4, 7, 6, 44, 47, 46, 74, 64, 67, 76, 77, 66, 476, 467, 746, 764, 647, 674; 444, 447, 446, 744, 644, 474, 464; 777, 774, 776, 477, 677, 747, 767; 666, 664, 667, 466, 766, 646, 676.

15. How many numbers of 1, 2, 3, 4, and 5 digits can be formed out of the number 56789, provided, that no digit occurs more than once in the connection with exactly the same digits?

Ans. 31. Numbers, 5, 6, 7, 8, 9; 56, 57, 58, 59, 67, 68, 69, 78, 79, 89; 567, 568, 569, 578, 579, 589, 678, 679, 689, 789; 5678, 5679, 5689, 5789, 6789; 56789.

16. Three travellers, A, B, C, on their journey come to three roads, and may either travel all together on one of the three, or singly, on the three roads, or two on the same road, and the third on a different road. How many selections may they make of their routes?

Ans. 27. Three, when all together; six, when they go separately; six, when A and B go together, and C goes by himself; six, when A and C are together, and B by himself; six, when B and C are together, and A by himself.

17. How many numbers composed of three digits each can be made out of the number 123456789, provided, that all the numbers thus formed shall differ by at least one digit. *Ans.* 84.

18. How many numbers of 4 digits each can be formed out of the same number, 123456789, so as to fulfil the above condition?

Ans. 126.

19. How many numbers composed of single digits, of 2 digits, of 3 digits, of 4 digits, and of 5 digits, can be formed out of the number 12345, in such a way that each number shall differ from all the other numbers by at least one digit.

Ans. 31. Numbers, 1, 2, 3, 4, 5; 12, 13, 14, 15, 23, 24, 25, 34, 35, 45; 123, 124, 125, 134, 135, 145, 234, 235, 245, 345; 1234, 1235, 1345, 1452, 2345; 12345

20. How many numbers composed of 1, 2, 3, 4, and 5 digits can be formed out of the number 12345, in such a way that the same digit may be repeated once, twice, &c., in the same number, and the several numbers necessarily differing only in the position of the digits.

Ans. 3905. Numbers, 1, 2, 3, 4, 5; 12, 13, 14, 15, 21, 31, 41, 51, 23, 24, 25, 32, 42, 52, 34, 35, 43, 53, 45, 54, 11, 22, 33, 44, 55; 111, 112, 113, 114, 115, 211, 311, 411, 511, 121, 131, 141, 151, &c.

21. A gentleman being asked what he would take for a valuable horse, replied, that his price was a cent for every change that he could make in the position of the 32 nails in the horse's shoes. What was his price?

Ans. 263130836933693530167218012160000 dollars.

LOGARITHMS.

403. The logarithm of a quantity is the exponent of the power to which it is necessary to raise an invariable quantity, called the *base*, to produce the quantity.

Let a be the invariable quantity or base, x its logarithm, and y the quantity given; then, $a^x = y$. In this equation, x is the logarithm of y. It is usual to write logarithm, log., or simply, l. Hence, $x = \log. y$, or $l.\ y$. We will begin the discussion by supposing $a > 1$, and x positive, and, though the general principles of logarithms are true for quantities as well as for numbers, yet, as in practice the logarithms of algebraic quantities are seldom used, we will first show the relations between numbers and their logarithms. Resuming the equation, $a^x = y$, we see that when $y = 1$, $x = 0$, for, $a^0 = 1$. Hence, log. 1 = 0, and, since this will be so, whatever may be the value of a, we see that the log. 1 is always zero. Every value given to x, above zero, will cause y to increase more and more; when $x = 1$, $y = a$. When x has values attributed to it greater than 1, y will exceed a more and more, until $x = \infty$ gives also $y = \infty$. Suppose, for instance, $a = 8$, then $x = 0$ and $x = 1$ give $y = 1$ and $y = 8$. Moreover, $x = 1, 2, 3, 4$, &c., gives $y = 8, 64, 512, 4096$, &c. Finally, $x = \infty$ gives $y = \infty$, for, $8^\infty = \infty$. It is plain that, when x has an intermediate value between 0 and 1, that y will have a value between 1 and 8, and that by making x pass through all possible positive values, entire and fractional, between 0 and 1, and 1 and ∞, y will be made to pass through all possible positive values between 1 and 8, and 8 and ∞. The contrary of what we have just seen will be the case when x is negative. For the equation, $a^{-x} = y$, will become $\frac{1}{a^x} = y$; and it is plain that, as x increases y will decrease, and that when $x = \infty$, $y = 0$. To illustrate this, let $a = 8$, and let $x = 0, 1, 2, 3, 4$, &c.,

then, $y = 1, \frac{1}{8}, \frac{1}{64}, \frac{1}{512}, \frac{1}{4096}$, &c. And it is evident that, by giving to x all possible positive values, entire and fractional, between 0 and ∞, y may be made to pass through all possible positive values between 1 and 0. Suppose now, $a < 1$. It will still be true that all possible values may be formed from the powers of one number, but the order of the numbers will be reversed. For values of x, between the limits 0 and $-\infty$, we will get all possible positive numbers between 1 and $+\infty$; and for all values of x, between 0 and $+\infty$, we will get all possible positive numbers between 1 and 0.

The remarkable fact that all positive numbers might be regarded as the powers of one invariable number, led to the invention of logarithmic tables by Napier, for the purpose of abridging complicated numerical calculations. The invariable number is called *the base of the system* of logarithms, and may evidently be any number whatever, except unity. Since all the powers of unity are unity, it is plain that this number cannot be taken as a base. The base of the common system of logarithms is 10. The powers of this number are more readily formed than the powers of any other number whatever, and this was the main reason for its selection.

We have $(10)^0 = 1$,	Hence, log. $1 = 0$,
$(10)^1 = 10$,	" log. $10 = 1$,
$(10)^2 = 100$,	" log. $100 = 2$,
$(10)^3 = 1000$,	" log. $1000 = 3$,
&c.	&c.

We see that, as the number changes (the base being the same), the logarithm also changes. It is plain, moreover, that any change in the base, the number being unaltered, will produce a change in the logarithm. Thus, if the base is 12, since $12^1 = 12$, we have log. $12 = 1$. The logarithm of 10 in this system will then not be 1, as in the system whose base is 10, but will be some fractional number less than 1. Hence, we see that logarithms depend upon the number and upon the base. That part of the logarithm in any system which depends upon the base is called *the modulus of the system*. A *table of logarithms* is a table exhibiting the logarithms of all numbers between certain limits, as 0 and 100, or 0 and 10000, calculated to a particular base.

The general properties of logarithms are independent of any particular base. A few of these general properties will now be demonstrated.

First Property.

404. The logarithm of the product of any number of factors taken in the same system, is equal to the sum of the logarithms of those factors.

For, let a represent the base of the system, y, y', y'', y''', &c., several numbers, and x, x', x'', x''', &c., their corresponding logarithms. Then, $a^x = y$, $a^{x'} = y'$, $a^{x''} = y''$, $a^{x'''} = y'''$, &c. Multiplying these equations together, member by member, we have $a^{x+x'+x''+x'''+\text{\&c.}} = yy'y''y'''$, &c. Then, since the exponent of the base is (from the definition) the logarithm of the second member, we have $x + x' + x'' + x''' +$ &c. $=$ log. $yy'y''y'''$, &c. But the first equations give $x =$ log. y, $x' =$ log. y', $x'' =$ log. y'', &c. Hence, log. $y +$ log. $y' +$ log. $y'' +$ log y''' $+$ &c. $=$ log $yy'y''y'''$ &c., as enunciated. Then, since the logarithm of the product of any number of factors is equal to the sum of the logarithms of these factors, it follows that to multiply by means of logarithms, we have only to add together the logarithms of the factors, and to find the number corresponding to this sum. This number will be the product required. Thus, let it be required to multiply 100 by 10, by means of logarithms. The log of 100 is 2, in a system whose base is 10; the log of 10 is 1; the sum of these logarithms, 3, is the logarithm of the product. Look for the number corresponding to 3 as a logarithm, and you will have 1000 as the product required. To exhibit the whole in an equation, we have, log. $100 \times 10 =$ log. $100 +$ log. 10 $= 2 + 1 = 3$; the number corresponding to which is 1000.

We have, then, for multiplication by means of logarithms, this

RULE.

Add together the logarithms of the several factors. This sum will be equal to the logarithm of the product. Look in the table of logarithms for the number corresponding to this logarithm, and you will have the product required.

EXAMPLES.

1. Required the equivalent expression to log. $a\ b\ c\ d$.

Ans. Log. $a +$ log. $b +$ log. $c +$ log. d.

2. Required the equivalent expression to log. $6an$.

Ans. Log. $6 +$ log. $a +$ log. n.

3. Required the equivalent expression to log. $(a + b)^n c^m d^p$.

Ans. Log. $(a + b)^n$ + log. c^m + log. d^p.

Second Property.

405. The logarithm of the quotient arising from dividing one quantity by another is equal to the logarithm of the dividend minus the logarithm of the divisor. For, let x equal the quotient of two quantities, m and n, then $x = \frac{m}{n}$, or $nx = m$. And, since, when two quantities are equal their logarithms must be equal, we have log. nx = log. m, or log. n + log. x = log. m. Hence, log. x = log. m — log. n, as enunciated. From this we derive for division, by means of logarithms, the following

RULE.

From the logarithm of the dividend subtract the logarithm of the divisor; the remainder will be the logarithm of the quotient. The number in the table corresponding to this logarithm will be the quotient required.

Thus, to find the quotient of 1000 by 10, we have log. 1000 — log. 10 $= 3 - 1 = 2$, and the number corresponding to 2 as a logarithm is 100.

EXAMPLES.

1. Find the equivalent expression to log. $\frac{am}{bn}$.

Ans. Log. am — log. bn, or log. a + log. m — log. b — log. n.

2. Find the equivalent expression to log. $\frac{xy}{b + n}$.

Ans. Log. x + log. y — log. $(b + n)$.

3. Find the equivalent expression to log. $\frac{x + y}{bn}$.

Ans. Log. $(x + y)$ — log. b — log. n.

4. Find the equivalent expression to log. $\frac{xy}{bn}$.

Ans. Log. x + log. y — log. b — log. n.

5. Find the equivalent expression to log. $\left(\frac{xy}{bn}\right)^p$.

Ans. Log. $(xy)^p$ — log. $(bn)^p$.

6. Find the equivalent expression to log. $\left(\frac{a+b}{c}\right)$.

Ans. Log. $(a+b)$ — log. c.

Third Property,

406. The logarithm of the power of a number is equal to the exponent of the power into the logarithm of the number.

For, take the equation, $a^x = y$, and raise both members to the m^{th} power, we will have $a^{mx} = y^m$ (A). In equation (A), m is any number whatever, positive or negative, entire or fractional. But, since the exponent of the base is the logarithm of the second member, equation (A) gives $mx =$ log. y^m; and the first equation gives $x =$ log. y. Hence, m log. $y =$ log. y^m, or log $y^m = m$ log. y, as enunciated. It follows, then, that to raise a number to a power by means of logarithms, we have only to apply this

RULE.

Multiply the logarithm of the number, as found in the table, by the exponent of the power to which it is to be raised. This product will be the logarithm of the power, and the number found in the table corresponding to this logarithm will be the required power.

Thus, to raise 10 to the second power by means of logarithms, we multiply the logarithm of 10, which is 1, by 2, the exponent of the power. The product, 2, is the logarithm of the power, and the number corresponding to this logarithm is 100. So that we have log. $(10)^2 =$ 2 log. $10 = 2$; the number corresponding to which is 100.

EXAMPLES.

1. Find the equivalent expression to log. $(a+b)^m$.

Ans. m log. $(a+b)$.

2. Find the equivalent expression to log. $c^n(a+b)^m$.

Ans. n log. $c + m$ log. $(a+b)$.

The 1st. property is here used in connection with the 3d.

3. Find the equivalent expression to log. $a^n (mr)^n$.

Ans. n (log. a + log. r + log. m).

4. Find the equivalent expression to log. $a^n \left(\frac{b}{c}\right)^n$.

Ans. n (log. a + log. b — log. c).

The 2d. and 3d. properties both employed.

5. Find the equivalent expression to log. $\frac{(a+b)^m}{h^2}$.

Ans. m log. $(a+b)$ — 2 log. h.

6. Find the equivalent expression to log. $\frac{a^2 - x^2}{(a+x)^3}$.

Ans. Log. $(a-x)$ — 2 log. $(a+x)$.

7. Find the equivalent expression to log. $\frac{(a+x)^3}{a^2 - x^2}$.

Ans. 2 log. $(a+x)$ — log. $(a-x)$.

Fourth Property.

407. The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.

For, take the equation, $a^x = y$, and extract the m^{th} root of both members. Then, $\sqrt[m]{a^x} = \sqrt[m]{y}$, or, $a^{\frac{x}{m}} = \sqrt[m]{y}$. Hence, $\frac{x}{m} = \log. \sqrt[m]{y}$ (A.), the exponent of the base being the logarithm of the second member. But the first equation gives $x = \log. y$; equation (A) then becomes log. $\frac{y}{m} = \log. \sqrt[m]{y}$, or log. $\sqrt[m]{y} = \log. \frac{y}{m}$, as enunciated.

For extracting roots by means of logarithms we have, therefore, this

RULE.

Divide the logarithm of the number, whose root is to be extracted, by the index of the root. The quotient will be the logarithm of the root; the number found in the table, corresponding to this logarithm, will be the root required.

Thus, to extract the square root of 100 by means of logarithms, we divide 2, the logarithm of 100, by 2, the index of the root. The quotient, 1, is the logarithm of the root, and 10, the number corresponding to this logarithm, is the root required.

EXAMPLES.

1. Find the equivalent expression to log. $\sqrt[n]{x+y}$.

Ans. $\frac{\text{Log.}\ (x+y)}{n}$.

2. Find the equivalent expression to log. $\sqrt{a^2 - x^2}$.

Ans. $\frac{\text{Log.}\ (a^2 - x^2)}{2} = \frac{\text{log.}\ (a+x)\ (a-x)}{2} = \frac{1}{2}\ \text{log.}\ (a+x) + \frac{1}{2}\ \text{log.}\ (a-x)$.

3. Find the equivalent expression to log. $a^2\ \sqrt[3]{a^5}$.

Ans. $\frac{11}{3}$ log. a.

4. Find the equivalent expression to log. $\sqrt[3]{(a+b)^2}$.

Ans. $\frac{2}{3}$ log. $(a+b)$.

408. Since, $a^0 = 1$, it matters not what is the value of a, and, since the equation, $a^0 = 1$, gives $0 = \text{log. } 1$, it follows that the logarithm of unity in every system of logarithms is equal to zero. From this we readily deduce the

Fifth Property.

The logarithm of the reciprocal of a number is equal to the logarithm of the number taken negatively.

For, let N be the number, then $\frac{1}{\text{N}}$ will be the reciprocal of the number, and we have log. $\frac{1}{\text{N}} = \text{log. } 1 - \text{log. N} = 0 - \text{log. N} = -\text{log. N}$, as enunciated.

EXAMPLES.

1. Required the equivalent expressions to log. $\frac{(a+x)^2}{x^3}$, and log. $\frac{x^3}{(a+x)^2}$.

Ans. 2 log. $(a+x) - 3$ log. x, and 3 log. $x - 2$ log. $(a+x)$.

2. Required equivalent expressions to log. $\frac{(a^2 - x^2)}{(a+x)^4}$, and log. $\frac{(a+x)^4}{a^2 - x^2}$.

Ans. Log. $(a-x)-3$ log. $(a+x)$, and 3 log. $(a+x)-$log. $(a-x)$.

Corollary.

409. When $a > 1$, we have $a^\infty = \infty$. Hence, $\infty =$ log. ∞. That is, the logarithm of infinity in a system, whose base is greater than unity, is, itself, infinity. And, since $0 = \frac{1}{\infty}$, it follows from the fifth property, that log. $0 = -\infty$. That is, the logarithm of zero in a system, whose base is greater than unity, is minus infinity.

When $a < \frac{1}{10}$, a number, $\frac{1}{10}$ for example, it is plain that the base must be raised to a power denoted by $-\infty$, in order to produce infinity. Thus, $(\frac{1}{10})^{-\infty} = \frac{1}{(\frac{1}{10})^\infty} = (10)^\infty = \infty$. Hence, the logarithm of infinity in a system, whose base is less than unity, is minus infinity. Hence, by the fifth property, the logarithm of zero in a system whose base is less than 1, is plus infinity. Thus, it is plain that $(\frac{1}{10})^\infty = 0$, from which $\infty =$ log. 0.

Sixth Property.

410. The logarithm of any base, taken in its own system, is unity.

For, suppose there are any number of bases, a, b, c, &c., and designate the logarithms in the systems, of which these are the bases, by log. log.′ log.″ &c. We have $a^1 = a$, $b^1 = b$, $c^1 = c$, &c. Hence, $1 =$ log. a, $1 =$ log.′ b, $1 =$ log.′ c, &c.

Thus, 1 is the logarithm of 10 in a system whose base is 10. But, in the system whose base is 8, the logarithm of 10 must be greater than 1. Because, $8^1 = 8$, or $1 =$ log. 8. Hence, a number greater than 1 must be the logarithm of 10.

Seventh Principle.

411. If we have a table of logarithms calculated to a particular base, the logarithms of the numbers in this table, divided by the logarithm of a second base, taken in the first system, will give, as quotients, the logarithms of the same numbers in the second system.

For, let $a^x = y$, and $b^z = y$. Then $a^x = b^z$ (A). Distinguish the logarithms of the two systems by log. and log.'. Taking the log. of both members of (A), we have x log. $a = z$ log. b. But, since $x =$ log. y, and $z =$ log'. y, and log. $a = 1$, this equation becomes log. $y =$ log.' y, log. b. Then $\text{log.}' y = \frac{\text{log. } y}{\text{log. } b}$, as enunciated.

412. The two systems in most common use are the Napierian, whose base is 2.718281828, and the common, whose base is 10. Suppose the Napierian logarithms to be designated by log.', and the common by log. Then, if we knew the common logarithm of any number, we can get the Napierian logarithm of the same number, by dividing the common logarithm of the number by the common logarithm of the Napierian base; or we can get the common logarithm of a number, knowing its Napierian logarithm, by multiplying the Napierian logarithm by the common logarithm of the Napierian base.

DIFFERENTIAL OF a^x.

413. The Binomial Formula enables us to find the differential of a^x. For, let $u = a^x$, and give x an increment, h. Then, $u' = a^{x+h} = a^x a^h$, and $u' - u = a^x(a^h - 1)$. Let $a = 1 + b$, and develop a^h by the binomial formula, we will have $a^h = (1 + b)^h = 1 + hb + \frac{h(h-1)b^2}{1 \,.\, 2}$ $+ \frac{h(h-1)(h-2)b^3}{1 \,.\, 2 \,.\, 3} + \text{\&c.} = 1 + h\left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \text{\&c.}\right)$, by neglecting the higher powers of h, as infinitely small quantities of the second order and higher orders. But, since $b = a - 1$, we have $a^h = 1 + h\left((a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.}\right) = 1 + \text{N}h$, by designating the quantity within the parenthesis by N. Hence, $u' - u = a^x(a^h - 1) = a^x\text{N}h$, and $du = a^x\text{N}dx = u\text{N}dx$; from which $dx = \frac{1}{\text{N}} \cdot \frac{du}{u}$. And, since $u = a^x$, we have $x =$ log. u, and $dx = d(\text{log. } u)$. Moreover, since a is the base, and N depends entirely upon a, we see that the differential of a logarithm is equal to the reciprocal of a constant, dependent upon the base into the differential of the quantity divided by the quantity.

EXAMPLES.

1 Required differential of b^y. *Ans.* $Nb^y dy$.

2. Required differential of log. $(1 + x)$.

$$Ans.\ \frac{1}{N} \cdot \frac{d(1+x)}{1+x} = \frac{1}{N} \cdot \frac{dx}{1+x}.$$

3. Required the differential of log. $(1 + x)^2$.

$$Ans.\ \frac{1}{N} \cdot \frac{2(1+x)dx}{(1+x)^2} = \frac{2}{N} \cdot \frac{dx}{(1+x)}.$$

LOGARITHMIC SERIES.

414. A logarithmic series is one which will enable us to calculate approximatively the logarithms of any number whatever. We have seen that the logarithm of a number depends, 1st, on the number; 2d, on the base. Hence, the development of a logarithm must contain the number, or some quantity dependent upon the number, and some quantity dependent upon the base. These two facts serve as a guide in assuming the form of development. Suppose we have the equation, $a^y = x$, in which x is the number, and y the logarithm. Let us assume log. $x = A + A'x + A''x^2 + A'''x^3 +$ &c., in which A, A′, A″, &c., are independent of x, and dependent upon the base, a. Now, if we make $x = 0$, the first member becomes infinite, whilst the second reduces to a finite quantity, A. The assumed form is then wrong. Again, assume log. $x = Ax + A'x + A''x^2 +$ &c. Making $x = 0$, we get $\pm \infty = 0$, which is absurd. The development then cannot be made under the second form. An examination of the two forms shows that we have assumed the logarithm to be developed in the powers of the number, and we have seen that it cannot be so developed. Since log. x cannot be directly developed, let us see whether log. $(1 + x)$ can be expanded into a series. Assume log. $(1 + x) = Ax + A'x^2 + A''x^3 + A'''x^4 +$ &c. (B). When $x = 0$ we have log. $1 = 0$, which presents no absurdity. Taking the differentials of both members of (B), and dividing out by dx, we have, $\frac{1}{N} \cdot \frac{1}{1+x} = A + 2A'x + 3A''x^2 + 4A'''x^3 +$ &c. (C), in which, $N = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3}$ — &c. Making $x = 0$ in (C), we get $A = \frac{1}{N}$. Differentiating (C),

we get, after dividing out by dx, $-\frac{1}{N}\cdot\frac{1}{(1+x)^2} = 2A' + 2\,.\,3A''x + 3\,.\,4\,A'''x^2 + \&c.$ (D), whence $A' = -\frac{1}{2N}$. Differentiating (D), and dividing out by dx, we have $+\frac{2}{N}\cdot\frac{1}{(1+x)^3} = 2\,.\,3A'' + 2\,.\,3\,.\,4A'''x + \&c.$ (E). Whence, by making $x = 0$, there results, $A'' = +\frac{1}{3N}$. The law for finding the coefficients is now plain; A''' will be found equal to $-\frac{1}{4N}$, $A^{iv} = \frac{+1}{5N}$, $A^{v} = -\frac{1}{6N}$, &c. Replacing the constants in (B) by their values, we have $\log.\,(1+x) = \frac{1}{N}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.\right)$. (P.)

Since $(1 + x)$, the number whose logarithm is developed, has been expanded in the powers of x; we conclude that, though the logarithm of a number cannot be developed in the powers of the number, it may be in the powers of a number less by unity.

Since $N = (a-1) - \frac{(a-1)}{2} + \frac{(a-1)}{3}$, &c., in which a is the base of the system, it is plain that the factor, $\frac{1}{N}$, in equation (P), depends for its value upon the base alone. Equation (P) shows, moreover, that the logarithm of a number is composed of two factors, one dependent upon the base alone, and the other dependent upon the number alone. The factor which depends upon the base for its value is called *the modulus of the system*. The modulus is usually represented by M, or M′.

415. If we take the logarithm of $(1 + x)$, in a system whose base is a', and denote the new logarithm by log.′, it is plain that the new development will differ from the first only in this notation and in the modulus. Hence, we will have

$$\log.\,(1+x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.\right), \text{ and}$$

$$\log.'\,(1+x) = M'\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.\right).$$

Hence, $\log.\,(1+x) : \log.'\,(1+x) :: M : M'$.

That is, the logarithms of the same number in two different systems are to each other as the moduli of these systems.

416. Since the modulus is expressed in terms of the base, it follows that if the base be given, the modulus can be determined, and conversely, the base can be found from the modulus.

If, then, we make $M' = 1$, the base of the system will become known. The system of logarithms, whose modulus is unity, is called Napierian, from Lord Napier, the inventor of logarithms. The base of the system, calculated from the assumed modulus, is 2·718281828.

COMMON AND OTHER LOGARITHMS FOUND FROM NAPIERIAN.

417. The proportion, $\log. (1+x) : \log'. (1+x) :: M : M'$, becomes, when $M' = 1$, $\log. (1+x) : \log'. (1+x) : M : 1$. Hence, we have $\log. (1+x) = M \log'. (1+x)$ (A).

This equation shows *that the Napierian logarithm of any number, multiplied by the modulus of any other system, will give the logarithm of the same number in that system. Moreover, equation* (A), *compared with the 7th Property, shows that the modulus of any system is equal to the logarithm of the Napierian base of that system.*

MEASURE OF ANY MODULUS.

418. Let a be the base of any system, and let the logarithms of this system be designated by log., and those of the Napierian by log′. Let $(1+x)$ be any number, then the 7th Property will give $\log. (1+x) = \frac{1}{\log.' a} \log'. (1+x)$ (B). Equation (B) compared with (A) shows that $M = \frac{1}{\log'. a}$. *Hence, the measure of the modulus of any system is equal to the reciprocal of the Napierian logarithm of the base of that system.*

This measure is even true for the Napiarian modulus itself, for we have $M' = \frac{1}{\log'. e} = \frac{1}{1} = 1$.

TABLE OF NAPIERIAN LOGARITHMS.

419. The formula, $l(1+x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.,\right)$

becomes, when $M = 1$, $l'(1 + x) = (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$ $+$ &c.,) (C).

Making $x = 1$, we have, $l'2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c.

This series enables us to calculate the Napierian logarithm of 2, and, by making $x = 2, 3, 4, 5$, &c., we can calculate the Napierian logarithm of 3, 4, 5, 6, &c. But the series does not converge rapidly enough, because, for every number greater than 2, the series goes on increasing continually, and it would be necessary to take an infinite number of terms in order to make the calculation in any degree correct. A simple artifice will enable us to convert the above series into a converging one, that is, into a series in which the terms will become smaller and smaller; so that all after a certain number may be neglected without affecting the result materially. Two transformations are used to affect this convergence.

First Transformation.

420. Make $x = \frac{1}{y}$ in equation (C), it will become

$$l'\left(1 + \frac{1}{y}\right) = l'\left(\frac{y+1}{y}\right) = l'(1 + y) - l'y = \frac{1}{y} - \frac{1}{2y^2} + \frac{1}{3y^3} - \frac{1}{4y^4} + \text{\&c., (D).}$$

This series will become more converging as y increases.

Making $y = 1, 2, 3, 4, 5, 6$, &c., in succession, we get

$$l'2 - l'1 = l'2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{\&c.,}$$
$$l'3 = l'2 + \tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \tfrac{1}{165} - \text{\&c.,}$$
$$l'4 = l'3 + \tfrac{1}{3} - \tfrac{1}{18} + \tfrac{1}{81} - \tfrac{1}{324} + \tfrac{1}{1215} - \text{\&c.,}$$
$$l'5 = l'4 + \tfrac{1}{4} - \tfrac{1}{32} + \tfrac{1}{192} - \tfrac{1}{1024} + \tfrac{1}{5120} - \text{\&c.}$$

The first series enables us to calculate the Napierian logarithm of 2. The second series enables us to calculate the Napierian logarithm of 3 when that of 2 is known; and the formula, evidently, only can be used when the logarithm of the next lowest number is known.

Second Transformation.

421. The preceding formula does not give a sufficiently converging series for small numbers. A better series can be obtained in the fol-

lowing manner. In the equation, $l'(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$, we get, by changing $+x$ into $-x$, $l'(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \&c.$ Subtracting the second series from the first, we get $l'(1+x) - l'(1-x) = l'\left(\frac{1+x}{1-x}\right) = 2\,(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \&c.,)$ (E). Place $\frac{1+x}{1-x} = 1 + \frac{1}{z}$, in which z is a whole number. Then, $x = \frac{1}{2z+1}$, and replacing x in (E) by this value, we get $l'(1 + \frac{1}{z})$, or $l'(1+z) - l'z = 2\left(\frac{1}{2z+1} + \frac{1}{3\,(2z+1)^3} + \frac{1}{5\,(2z+1)^5} + \frac{1}{(7\,2z+1)^7} + \&c\right)$. This series converges more rapidly than (D), and will answer even for small numbers.

Let $z = 1, 2, 3, 4, 5, 6, \&c.$, we will have

$$l'2 - l'1 = l'2 = 2\left(\tfrac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5\,(3)^5} + \frac{1}{7\,(3)^7} + \&c.,\right)$$

$$l'3 - l'2 = 2\left(\tfrac{1}{5} + \frac{1}{3\,(5)^3} + \frac{1}{5\,(5)^5} + \frac{1}{7\,(5)^7} + \frac{1}{9\,(5)^9} + \&c.\right.$$

$$l'4 - l'3 = 2\left(\tfrac{1}{7} + \frac{1}{3\,(7)^3} + \frac{1}{5\,(7)^5} + \frac{1}{7\,(7)^7} + \frac{1}{9\,(7)^9} + \&c.,\right.$$

$$l'5 - l'4 = 2\left(\tfrac{1}{9} + \frac{1}{3\,(9)^3} + \frac{1}{5\,(9)^5} + \frac{1}{7\,(9)^7} + \frac{1}{9\,(9)^9} + \&c.,\right)$$

$$l'6 - l'5 = 2\left(\tfrac{1}{11} + \frac{1}{3(11)} + \frac{1}{5\,(11)^5} + \frac{1}{7\,(11)^7} + \frac{1}{9\,(11)^9} + \frac{1}{11\,(11)^{11}} + \&c.\right)$$

Remarks.

1. For small numbers it is necessary to take a good many terms of this series, but when the numbers are large one or two terms will answer. Thus, let $x = 1000$, then $l'1001 - l'1000 = 2\left(\frac{1}{2001} + \frac{1}{3(2001)^3} + \frac{1}{5(2001)^5} + \&c\right)$. We see that the first term of this series will only add about $\frac{1}{1000}$ to the logarithm of 1000, and the second term will add less than $\frac{1}{6000000000000}$. And, since logarithms are seldom

carried further than the 7th place of decimals, all the terms after the first may be neglected.

2. In constructing a table it is only necessary to calculate the logarithms of prime numbers, since the logarithm of any number made up of factors is equal to the sum of the logarithms of its factors. Thus, $l'10 = l'2 + l'5$, $l'12 = l'3 + l'4 = l'3 + 2l'2$, &c.

3. Knowing the Napierian logarithm of 10, we can find the modulus of the common system. For, log.′ 10 : log. 10 : : 1 : M. Hence, $M = \frac{\log. 10}{l'10}$; but log. 10 = 1, and the calculated value of log.′ 10 is 2.302585093. Therefore, $M = \frac{1}{2{\cdot}302585093} = 0{\cdot}434294482$.

If we now multiply the Napierian logarithms found from the series by this modulus, we will have a table of common logarithms.

ADVANTAGES OF THE COMMON SYSTEM.

422. 1. *A fixed law for the characteristic.*

The characteristic of a logarithm is its entire part. Thus, 2 is the characteristic in the logarithm 2·302585093.

Whatever may be the base of the system, the logarithms of the powers of that base will contain no decimals, and, therefore, be characteristics only. Thus, let 7 be the base, then the log. 7 = 1, log. 49 = 2, &c.

In the common system, with the base 10, we have log. 10 = 1, log. 100 = 2, log. 1000 = 3, &c., and we see that the characteristic is always one less than the number of figure places in the number. This is true, whether the number be an exact power of the base or not; thus, 512, lying between 100, whose logarithm is 2, and 1000, whose logarithm is 3, has for its logarithm 2 and a certain decimal. Now, 10 is the only number in the whole range of numbers, which, taken as a base, will give logarithms whose characteristics are formed according to a fixed law.

In a table of common logarithms, the characteristics are not written, *since they are always one less than the number of figures in the given numbers.*

2. *A table of logarithms for decimals need not be constructed.*

$$\text{Since, Log. } {\cdot}1 = \log. \tfrac{1}{10} = \log. 1 - \log. 10 = -1.$$
$$\text{Log. } {\cdot}01 = \log. \tfrac{1}{100} = \log. 1 - \log. 100 = -2.$$
$$\text{Log. } {\cdot}001 = \log. \tfrac{1}{1000} = \log. 1 - \log. 1000 = -3.$$

And, since, in general, log. $\frac{1}{(10)^m}$ = log. 1 — m log. 10 = — m, it is plain that the characteristic of a decimal is negative, and one greater than the number of ciphers between the decimal point and the first significant figure. This is true, also, when the decimal is not the reciprocal of some power of the base. Thus, log. ·08 = log. $\frac{8}{10}$ = log. 8 — log. 10 = a decimal — unity. Hence, the log. ·08 must have a negative characteristic equal to minus unity. Moreover, if we have a decimal, such as ·00278 = $\frac{278}{100000}$, we will have log. ·00278 = log. 278 — log. 100000. And, since log. 278 is 2 whole number, plus a certain decimal; and, since the log. 100000 is 5, the log ·00278 = — 3, whole number plus a certain decimal. We see, from this example, that the logarithm of a decimal is obtained by regarding the decimal as a whole number, and prefixing a negative characteristic, one greater than the number of ciphers, between the decimal point and the first significant figure. The foregoing reasoning can be extended to any decimal whatever, because, change the decimal into an equivalent vulgar fraction, it will be seen that the number of figure places in the denominator exceed those in the numerator by one more than the number of ciphers between the decimal point and the first significant figure of the given decimal. Now, if the base were any other number whatever than 10, it is plain that a table of logarithms for whole numbers would not be applicable to decimals.

3. *The logarithm of a mixed decimal, such as* 42·733, *can be found from a table of logarithms constructed to the base* 10, *by regarding the mixed number as a whole number, and prefixing to the decimal part of the logarithm found in the table a characteristic one less than the number of figure places in the entire part of the mixed number.*

For, log. 42·733 = log. $\frac{42733}{1000}$ = log. 42733 — log. 1000 = 4 + a certain decimal — 3 = 1 + a decimal.

The foregoing rule for finding the logarithm of a mixed number can evidently be applied to any mixed number whatever.

4. The logarithms of numbers which differ only in the number of annexed ciphers, will differ only in their characteristics.

Thus, the logarithms of 125, 1250, 12500, and 125000, differ only in their characteristics. For, log. 1250 = log. 10 + log. 125 = 1 + log. 125; log. 12500 = log. 100 + log. 125 = 2 + log. 125; log. 125000 = log. 1000 + log. 125 = 3 + log. 125.

This property of common logarithms is very important, since it saves the trouble (as will be seen more fully hereafter) of constructing a table for numbers whose figure places exceed four.

APPLICATION OF COMMON LOGARITHMS.

423. The base of the common system being 10, the characteristic or entire part of the logarithms need not be written in the tables, since, as we have seen, it is always one less than the number of figure places in the given number. The decimal parts of the logarithms are then only written in the tables, and are generally carried as far as six places. The first vertical column on the left in the tables, contains the given numbers; the last vertical column, marked D, contains the tabular difference corresponding to two given logarithms, and is constructed by a method hereafter to be explained. The intermediate vertical columns, marked at top, 0, 1, 2, 3, &c., contain the decimal parts of logarithms.

The following table exhibits the logarithms and numbers between 100 and 130. When the given number contains four figure places, the

N	0	1	2	3	4	5	6	7	8	9	D
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	·4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	·300	·724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5368	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	·361	·775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	·095	·600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	·207	·602	·998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	·380	.766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	·320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	· ·38	·407	·706	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	·266	·626	·987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7718	8136	8490	8845	9198	9552	355
123	9905	·258	·611	·963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	· ·26	
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	·253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	332

first three will be found in the first vertical column on the left, and the fourth must be sought at the top of the page, among the numbers between 0 and 9, inclusive. Thus, 121 may be found in the first vertical column on the left, and the last figure of 1215 will be found in the column marked 5 at top. The logarithm of 121 is written just opposite to it in the column marked 0; 1210 will obviously have the same decimal part of its logarithm, but a characteristic greater by unity; the logarithm of 1215, on the other hand, must not only exceed the logarithm of 121 in characteristic, but also in decimal part, and its decimal part is then taken from the column marked 5.

TO FIND THE LOGARITHM OF A GIVEN NUMBER BELOW 10,000.

424. When the number is less than 100, the decimal part of the logarithm is found in the table just opposite the given number. The characteristic will be determined by the number of figure places in the number. For numbers above 100 and below 10,000, we have the following

RULE.

Look for the first three figures of the given number in the column marked N, *and if it contains but three, the decimal part of its logarithm will be found just opposite these three figures in the column marked* 0. *But if there are only four figures in this column, opposite the given number, the two left hand figures above, in the same column, where six figures are written, must be prefixed to the four figures found. Thus, the decimal part of the logarithm of* 110, *is* 041398, *and the entire logarithm,* 2·041398. *The decimal part of the logarithm of* 112 *is* 049218, *the first two figures being found opposite* 110. *The entire logarithm is then* 2·049218. *The logarithm of* 117 *is* 2·06816, *and of* 120, 2·079181.

425. When the given number contains four figures, look for the first three in the column marked N. The last four of the decimal part of its logarithm will be found just opposite, in the column marked at top by the fourth figure. To these four figures must be prefixed two decimals in the column marked zero, either found opposite the first three figures, or above, where six figures occur. Thus, to get the logarithm of 1213, we first find 121 in the first column, and then look just opposite, in the column marked 3, where we find 3861, and prefix to it 08, taken from the column marked 0. Hence, the logarithm of 1213 is 3·083861. The logarithm of 1283 is 3·108227, the two left hand figures of its decimal part being found above, where six figures occur.

When, however, we pass over a decimal with a point prefixed, the first two figures are to be found below, and not above. Thus, the logarithm of 1234 is 3·091315. In like manner, the logarithm of 1231 is 3·090258. The place of the point is always supplied by zero. If the left hand figures were not taken below, the logarithm of 1231, 1232, &c., would be less than the logarithm of 1230, 1229, &c.

EXAMPLES.

1. Required the logarithm of 129. *Ans.* 2·110590.
2. Required the logarithm of 1290. *Ans.* 3·110590.
3. Required the logarithm of 1297. *Ans.* 3·112940.
4. Required the logarithm of 1285. *Ans.* 3·108903.
5. Required the logarithm of 1233. *Ans.* 3·090963.
6. Required the logarithm of 1236. *Ans.* 3·092018.
7. Required the logarithm of 119. *Ans.* 2·075547.
8. Required the logarithm of 1191. *Ans.* 3·075912.

TO FIND THE LOGARITHM OF A NUMBER ABOVE 10,000.

426. Cut off on the right all the figures, except the first four of the highest denomination. Find the decimal part of these four left hand figures, as before, and prefix a characteristic, one less than the number of figure places in the given number previous to cutting off. Multiply the tabular difference by the figures cut off on the right, and from this product cut off as many places for decimals as there are figure places in the multiplier. Add the result to the logarithm before found. Thus, to find the logarithm of 129451, we cut off 51, and find the decimal part of 1294 to be 111934. To this we prefix a characteristic, 5, and we have 5·111934, which is truly the logarithm of 129400 (Art. 422). Next, we multiply the tabular difference, 335, by 51, and cut off 85 from the product, 17085; we would then have 170 to add to 5·111934; but, as ·85 is greater than ½, we increase 170 by 1, because 171 is nearer to 170·85 than is 170. Adding then 171, we have 5·112105 for the logarithm of 129451. In general, whenever the decimal cut off in the product, which results from multiplying the tabular difference by the right hand figures of the given number, exceeds ½, the last figure of the entire part of the product must be increased by 1.

The explanation of the process is simple. The tabular difference expresses the difference between the logarithms of consecutive numbers. Thus, the logarithm of 1293 is 3·1599; and of 1294 is 3·1934, and their difference, 335, is written in the column marked D. This column, it is plain, will moreover express the difference between the logarithms of numbers which differ by 10, 100, 1000, &c. We have seen that the logarithm of 129400 was 5·11934, but we were required to find the logarithm of 129451, and it remains to be seen how much the logarithm of 129400 is to be increased. We employ the principle, that the difference between any two numbers is to the difference of their logarithms, as the difference between any two other numbers is to their difference of logarithm. Hence, 129400 — 129300 : 335 : : 129451 — 129400 : x, or 100 : 335 : : 51 : x, in which x represents the augment to the logarithm of 129400, to give the logarithm of 129451. We find $x = \frac{17085}{100} = 170·85$.

EXAMPLES.

1\. Required the logarithm of 1309958. *Ans.* 6·117259.

In this example three places are cut off for decimals, because we took the difference between 1309000 and 1308000. The first term of the proportion was then 1000.

2\. Required the logarithm of 13080000. *Ans.* 7·116608.

3\. Required the logarithm of 13087654. *Ans.* 7·116862.

TO FIND A NUMBER CORRESPONDING TO A GIVEN LOGARITHM.

427\. When the decimal part of the logarithm can be found in the tables, the first three places of the required number will be found on the left, immediately opposite this logarithm, and in the column marked N. The fourth figure, which is to be annexed to the three already found, is to be taken from the number of the column in which the decimal part of the logarithm was written. Then point off, from the left of the number thus obtained, one more place for the entire part, than is indicated by the number of units in the characteristic. Thus, the number corresponding to the logarithm 2·107888 is 128·2, and to 1·107888 is 12·82. So, the number corresponding to the logarithm 3·088136 is 1225, and contains no decimal part.

But, when the decimal part cannot be exactly found, take the next less logarithm from the tables, and subtract this from the given logarithm. Annex two or more ciphers to this difference, and divide by the tabular difference. Annex the quotient to the number corresponding to the next less logarithm, and you will have the required number.

Thus, let it be required to find the number corresponding to the logarithm 4·1132800. The next less logarithm is 4·1132750, and the corresponding number 12980; annexing 4 ciphers to 50, the difference of logarithms, and dividing by 335, the tabular difference, we have a quotient, 149, which is to be annexed as a decimal to 12980. Hence, the number sought is 12980·149. It is plain that we may annex as many ciphers as we please to the difference of logarithms. The process just described being the reverse of the preceding, requires no explanation.

EXAMPLES.

1. Required the number whose logarithm is 4·07850.
Ans. 11981·18.

2. Required the number whose logarithm is 7·116608.
13080000.

3. Required the number whose logarithm is 8·075550.
Ans. 119000243·85.

4. Required the number whose logarithm is 3·017249.
Ans. 1040·52.

5. Required the number whose logarithm is 3·153209.
Ans. 1423·013.

Remarks.

When only one cipher has to be annexed to the difference between the given and next less logarithm, to make it divisible by the tabular difference, no cipher is prefixed to the quotient. But when, (as in example 5), two ciphers must be annexed to make the division possible, then one cipher must be prefixed to the quotient. And when, (as in example 3), three ciphers must be annexed, two must be prefixed to the quotient. And, in general, the number of ciphers prefixed is one less than the number of ciphers annexed, to make the division possible.

GENERAL EXAMPLES.

1. Required the product of 1·040, by 1·055 and 10·77.

Ans. 11·81687.

For log. 1·040 = 0·017033
log. 1·055 = 0·023252
log. 10·77 = 1·032216

And logarithm 1·072501 corresponds to 11·81687.

2. Required the quotient of 10·22 by 1·016. *Ans.* 10·059.

For log. 10·22 = 1·009451
and log. 1·016 = 0·006894
and their difference, 1·002557, corresponds to 10·059.

3. Raise 11·48 to the third power. *Ans.* 1512·95.

For log. 11·48 = 1·059942
3
3·179826 = log. 1512·95.

4. Required the thirty-second root of 1172. *Ans.* 1·21871.

For we have $\frac{\log. 1172}{32} = \frac{3{\cdot}068928}{32} = {\cdot}095904$.

And the corresponding number is 1·21871.

SOLUTION OF EXPONENTIAL EQUATIONS BY MEANS OF LOGARITHMS.

428. Suppose we have the exponential equal, $a^x = p$, then, x log. $a =$ log. p, and $x = \frac{\log. p}{\log. a}$; that is, the root is equal to the logarithm of p divided by the logarithm of a, and this root itself may be found by means of logarithms for log. $x =$ log. $\frac{\log. p}{\log. a} =$ log. log. p — log. log. a.

EXAMPLES.

1. Required the value of x in the equation, $10^x = 10$.

Ans. $x = 1$.

For, x log. 10 = log. 10, or x (1) = 1, or $x = 1$.

2. Required the value of x in the equation, $12^x = 13$.

Ans. 1·032.

For,	log. 13 = 1·113943	log. 1·113943 = 0·046862
and,	log. 12 = 1·079181	log. 1·079181 = 0·033094
		0·013768.

Hence, log. x = 0·013768, or x = 1·032.

3. Required the value of x in the equation, $100^x = 1000$.

Ans. $x = \frac{3}{2}$.

4. Required the value of x in the equation, $10^x = 125$.

Ans. x = 2·096910.

5. Required the value of x in the equation, $100^x = 125$.

Ans. x = 1·048455.

6. Required the value of x in the equation, $11^x = 11$.

Ans. $x = 1$.

For, log. x = log. log. 11 — log. log. 11 = log. 1·041393 — log. 1·041393 = 0·017614 — 0·017614 = 0. Now, since log. $x = 0$, x must be unity.

The result in this case might have been anticipated, but we have carried out the process to show its truthfulness.

ARITHMETICAL PROGRESSION.

429. Quantities, which increase or decrease by a common difference, are said to be in Arithmetical Progression.

Thus, 2, 4, 6, 8, 10, &c., constitute an arithmetical progression, the *common difference* being 2. In like manner, 10, 8, 6, 4, 2 constitute, also, an arithmetical progression, the common difference being 2. When, as in the first case, the common difference is an increment, the series is called *ascending;* and when, as in the second case, the common difference is a decrement, the series is called *descending*. The quantities, a, $a + d$, $a + 2d$, $a + 3d$, &c., constitute an ascending series; and a, $a - d$, $a - 2d$, $a - 3d$, constitute a descending series.

It is evident, likewise, that the quantities, $a + 3d$, $a + 2d$, $a + d$, a, form a descending series; and, in general, an ascending series, when read in reverse order, becomes a descending series, and conversely.

It is plain that the natural numbers, 1, 2, 3, 4, 5, 6, 7, &c., are in arithmetical progression, the common difference being 1.

It is plain that, if we know any term of an ascending series, we can find the succeeding term by adding to the known term the common difference. And, if we wish to find the term succeeding any known term of a decreasing series, we have only to subtract the common difference from the known term. In this manner, we may find any term of a series by a continued addition or subtraction of the common difference. But, as in a long series, this process would be tedious, it becomes necessary to deduce a formula by which any term, as the n^{th}, may be found without going through a tedious addition or subtraction.

FORMULA FOR THE *Nth* TERM.

430. The second term of the ascending series, a, $a + d$, $a + 2d$, &c., is $a + d$; that is, the common difference is added to the first term one less number of times than the place of the term. The third term is $a + 2d$; that is, the first term is increased by the common difference, taken once less than the place of the term. We discover the same law of formation in regard to the fourth term, and all succeeding terms. Hence, calling, the n^{th} term l, we have $l = a + (n - 1)d$; that is, *the* nth *term is equal to the first term, increased by the common difference, taken once less than the place of the term.*

If the series is decreasing, d is negative, and the formula becomes $l = a - (n - 1)d$. The first formula includes the second; and the first, therefore, is only necessary, care being taken to attribute to d its proper sign.

For an ascending series the first term is always the least, and for a descending series the first term is always the greatest.

The foregoing formula, $l = a + (n - 1)d$, may be used to obtain the last term, for n may represent any term whatever. When the common difference is equal to the first term, that is, $d = a$, then, $l = a + na - a = na$. The n^{th} term is then equal to the first term into the number of terms. When the number of terms is unity, that is, $n = 1$, then, $l = a$, as it ought to be.

EXAMPLES.

1. The first term of an ascending series is 2, the common difference 3, required the fifth term. *Ans.* 14.

2. Required the tenth term of the same series. *Ans.* 29.

3. Required the last term of an ascending series, of which the first term is 10, the common difference 5, and the number of terms 11. *Ans.* 60.

4. Required the eleventh term of a descending series, of which the first term is 60, and the common difference 5. *Ans.* 10.

5. Required the tenth term of an ascending series, of which the first term is 10, and the common difference 10. *Ans.* 100.

6. Required the last term of a decreasing series, of which the first term is 100, the common difference 10, and the number of terms 10. *Ans.* 10.

FORMULA FOR THE FIRST TERM.

431. The formula, $l = a + (n - 1)d$, contains four quantities, and, of course, if any three are given, the fourth can be determined. By transposition and reduction, we have $a = l - (n - 1)d$. That is, *the first term is equal to the last, or* n^{th} *term, minus the number of terms, less one, into the common difference.*

The formula is applicable both to an ascending and to a descending series, by attributing the proper sign to d.

EXAMPLES.

1. The fifth term of an ascending series is 14, and the common difference 3. Required the first term. *Ans.* 2.

2. The tenth term of the same series is 29. Required the first term. *Ans.* 2.

3. Required the first term of an ascending series, whose last term is 60, common difference 5, and the number of terms 11. *Ans.* 10.

4. Required the first term of a descending series, of which the last term is 10, the common difference 5, and the number of terms 11. *Ans.* 60.

FORMULA FOR THE COMMON DIFFERENCE.

432. From the formula, $l = a + (n - 1)d$, we get $d = \frac{l - a}{n - 1}$, that is, *the common difference is equal to the last term, minus the first term, divided by the number of terms less one.*

For a descending series, d is negative, and, by multiplying both members by minus unity, we get $d = \frac{a - l}{n - 1}$, that is, *the common difference of a descending series is equal to the first term, minus the last term, divided by the number of terms less one* When $l = a$, d will be zero, as it ought to be. When $n = 1$, $d = \infty$. The symbol of absurdity ought to appear under the hypothesis $n = 1$; for, when there is but one term, there can be no common difference, and therefore the assumption of its existence is absurd.

EXAMPLES.

1. The last term is 50, the first term 10, and the number of terms 3. Required the common difference. *Ans.* 20.

2. The last term is 10, the first term 50, and the number of terms 3. Required the common difference. *Ans.* 20.

3. The last term is 12, the first term 1, and the number of terms 4. Required the common difference. *Ans.* $\frac{11}{3}$.

FORMULA FOR THE NUMBER OF TERMS.

433. From $l = a + (n - 1)d$, we get $n = 1 + \frac{l - a}{d}$, that is, *the number of terms is equal to unity, added to the quotient arising from dividing the difference between the first and last terms by the common difference.*

The formula is applicable to a decreasing series, by attributing the appropriate sign to d; when $l = a$, n will be equal to 1, when $d = 0$, $n = \infty$.

EXAMPLES.

1. The last term is 50, the first term 10, and the common difference 20. Required the number of terms. *Ans.* 3.

2. The last term is 12, the first term 1, and the common difference $\frac{11}{3}$. Required the number of terms. *Ans.* 4.

3. The last term is 50, the first term 10, and the common difference 1. Required the number of terms. *Ans.* 41.

FORMULA FOR THE SUM OF THE TERMS.

434. We might get the sum of the terms by actually performing the addition; but, when the series contains many terms, this operation would be difficult, and we are enabled to deduce a formula which abridges the work. This formula is deduced from the remarkable property of an arithmetical progression, that the sum of any two terms at equal distance from the two extremes is equal to the sum of the two extremes. To show this, any term, as the m^{th}, counting from the first term, will be expressed by $a + (m - 1)d$, and the m^{th} term, counting from the right, will be expressed by $l - (m - 1)d$; and the sum of these terms is plainly $a + l$, a denoting the first term and l the last term.

Calling S the sum of the terms, we have $S' = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) \ldots$ to l, and reversing the series, $S = l + (l - d) + (l - 2d) + (l - 3d) \ldots$ to a. Adding these equations member by member, we get $2S = (a + l) + (a + l) + (a + l) +$ &c., up to n terms.

Hence, $2s = (a + l)n$, or $s = \frac{(a + l)n}{2}$, that is, *the sum of the terms is equal to the half sum of the first and last terms, into the number of terms.*

When $n = 2$, $s = a + l$, as it ought to be. When $l = a$, $s = na$, the common difference is then zero. When $n = 4$, $s = 2(a + l)$, as it ought to be. When $n = 0$, $s = 0$, as it ought to be.

The sum of the terms can be determined when a, l and n are known, or can be found from the data.

EXAMPLES.

1. Find the sum of the natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9. *Ans.* 45.

2. The first term of a series is 10, the last term 100, and the number of terms 5. Required the sum of the terms. *Ans.* 275.

3. The first term of an ascending series is 5, the common difference 5, and the number of terms 5. Required the sum of the terms.

Ans. 75.

4. The last term of an ascending series is 60, the number of terms 4, and the common difference 2. Required the sum of the terms.

Ans. 228.

5. The last term of a descending series is 60, the number of terms 4, and the common difference 2. Required the sum of the terms.

Ans. 252.

6. The last term of an ascending series is 50, the first term 6, and the common difference 2. Required the sum of the terms.

Ans. 644.

7. Find s, when a, d, and n, are known.

Ans. $S = \frac{1}{2} | 2a + (n - 1)d | n$.

8. The first term of an ascending series is 1, the common difference 1, and the number of terms 10. Required the sum of the terms. *Ans.* 55.

9. How many strokes does the common clock strike in 12 hours?

Ans. 78.

10. A body falling in vacuo will pass over about 16 feet the first second, 48 the next second, 80 the third second, &c. How far will it fall in 20 seconds, and what space will it pass over in the last second?

Ans. Entire space, 6400 feet; in last second, 624 feet.

11. A traveller goes 5 miles the first day, 15 the second, 25 the third, &c. Required the space that will have been passed over at the end of the tenth day. *Ans.* 500 miles.

12. Find the sum of the natural numbers, 1, 2, 3, &c., up to n terms.

Ans. $\frac{1}{2} n (n + 1)$.

13. Find the sum of the odd numbers, 1, 3, 5, 7, &c., up to n terms

Ans. n^2.

14. Find the sum of the even numbers, 2, 4, 6, 8, 10, &c., up to n terms. *Ans.* $n (n + 1)$.

15. Find the sum of the numbers, 6, 12, 18, 24, &c., up to n terms.

Ans. $3n (n + 1)$.

16. Find the sum of the odd numbers, 1, 2, 3, &c., up to 73.
Ans. 5329.

17. Find the sum of the even numbers, 2, 4, 6, &c., up to twenty-five terms. *Ans.* 650.

18. Find the sum of the numbers, 6, 12, 18, &c., up to the number 72. *Ans.* 468.

19. Find s, when l, d and n, are known.
Ans. $S = \frac{1}{2} | 2l - (n-1)d | n$.

TO FIND THE ARITHMETICAL MEAN.

435. The arithmetical mean between several quantities, is the quotient arising from dividing their sum by their number. The arithmetical mean between two quantities, m and n, is half their sum, that is, $\frac{1}{2}(m+n)$. We have seen that a geometric mean between two quantities is the square root of their product.

From the definition, the arithmetical mean between any number of terms in progression must be $\frac{S}{n} = \frac{\frac{(a+l)n}{2}}{n} = \frac{1}{2}(a+l)$, *that is, the arithmetical mean is equal to the half sum of the extremes.*

Corollary.

The first and last terms may be found when the arithmetical mean, the number of terms, and the common difference, are known.

For, from $S = \frac{(a+l)n}{2}$, we get $a = \frac{2S}{n} - (a + (n-1)d)$, or $a = 2M - a - (n-1)d$. Hence, $a = M - \frac{(n-1)d}{2}$.

And since $l = a + (n-1)d$, we get $l = M + \frac{(n-1)d}{2}$

The employment of the arithmetical mean frequently facilitates the solution of a problem in progression.

EXAMPLES.

1. The sum of four numbers in arithmetical progression is 20, and their continued product, 384. What are the numbers?

Ans. 2, 4, 6, 8.

For, $M = \frac{S}{n} = \frac{20}{4} = 5$. And the first term, $a = M - \frac{(n-1)d}{2}$ $= 5 - \frac{3}{2}d$; the second term, $5 - \frac{1}{2}d$; the third term, $5 + \frac{1}{2}d$; the fourth term, $5 + \frac{3}{2}d$. Hence, by the conditions of the problem, $(5 - \frac{3}{2}d)(5 - \frac{1}{2}d)(5 + \frac{1}{2}d)(5 + \frac{3}{2}d) = 384$, or $625 - \frac{250d^2}{4} + \frac{9d^4}{16} = 384$.

Making $d^2 = y$, we get, after reduction, $y^2 - \frac{1000y}{9} = -\frac{3856}{9}$. Hence, $y = \frac{500}{9} \pm \frac{464}{9} = +4$, or $+\frac{964}{9}$. The last value being rejected, we have $d^2 = +4$. Hence, $d = \pm 2$. The positive value will give $a = 5 - \frac{3}{2}d = 2$; and the second term, 4; the third, 6; the fourth, 8. The negative value of d will give the series in reverse order, and we see here a change of sign followed by a change of direction.

The same problem might be solved by making $2x$ the common difference, and $y - 3x$, the smaller extreme. Then the 4 terms will be represented by $y - 3x$, $y - x$, $y + x$, and $y + 3x$. And we have the two equations, $y - 3x + y - x + y + x + y + 3x = 4y = 20$, and $(y - 3x)(y + 3x)(y - x)(y + x) = (y^2 - 9x^2)(y^2 - x^2) = y^4 - 10x^2y^2 + 9x^4 = 384$. Substituting, in this last equation, the value of y drawn from the first, we get $625 - 250x^2 + 9x^4 = 384$. From which, $x = \pm 1$. Then $y - 3x = 5 - 3 = 2$, and $y - x = 4$, &c.

The student will readily perceive the advantage of representing the first extreme by $y - 3x$, and the common difference by $2x$.

2. The sum of 5 numbers in arithmetical progression is 15, and their product 120. What are the numbers? *Ans.* 1, 2, 3, 4, 5.

INSERTION OF MEANS BETWEEN THE EXTREMES OF A PROPORTION.

436. The formula, $d = \frac{l - a}{n - 1}$, enables us to insert any number of means, when the extremes are known. Let it be required to insert 3 means between 1 and 9. Then $a = 1$, and $l = 9$, and, since 3 means are to be introduced between 1 and 9, n must be 5. Hence, $d = \frac{9 - 1}{5 - 1} = 2$. Knowing the common difference, the terms succeeding the first can be readily formed. They are 3, 5, 7; and the 5 terms of the proportion are 1, 3, 5, 7, 9.

EXAMPLES.

1. Required 15 means between 1 and 9.

Ans. $d = \frac{1}{2}$. Series, $1 \,.\, 1\frac{1}{2} \,.\, 2 \,.\, 2\frac{1}{2} \,.\, 3 \,.\, 3\frac{1}{2} \,.\, 4 \,.\, 4\frac{1}{2} \,.\, 5 \,.\, 5\frac{1}{2} \,.\, 6 \,.\, 6\frac{1}{2} \,.\, 7 \,.\, 7\frac{1}{2} \,.\, 8 \,.\, 8\frac{1}{2} \,.\, 9$.

It is plain that an infinite number of means may be inserted between 1 and 9, by making the common difference indefinitely small.

2. Insert 10 means between 1 and 2.

Ans. $d = \frac{1}{11}$. Series, $1 \,.\, 1\frac{1}{11} \,.\, 1\frac{2}{11} \,.\, 1\frac{3}{11} \,.\, 1\frac{4}{11} \,.\, 1\frac{5}{11} \,.\, 1\frac{6}{11} \,.\, 1\frac{7}{11} \,.\, 1\frac{8}{11} \,.\, 1\frac{9}{11} \,.\, 1\frac{10}{11} \,.\, 2$.

GENERAL EXAMPLES.

1. Find 4 numbers in arithmetical progression, whose sum is 14, and the sum of whose squares is 54. *Ans.* 2, 3, 4, 5.

Let $y - 3x =$ lesser extreme, and let $2x =$ common difference. Then the other terms are $y - x$, $y + x$ and $y + 3x$. Hence, by the conditions, $y - 3x + y - x + y + x + y + 3x = 4y = 14$, or $y = 3\frac{1}{2}$. And $(y - 3x)^2 + (y - x)^2 + (y + x)^2 + (y + 3x)^2 = 4y^2 + 20x^2 = 54$, and eliminating y^2, we have $49 + 20x^2 = 54$, or $x^2 = \frac{1}{4}$. Then, $x = \pm \frac{1}{2}$. From which the above series.

2. Find 5 numbers in arithmetical progression, whose sum is 40, and the sum of whose squares is 410. *Ans.* 2, 5, 8, 11, 14.

Let $y - 4x =$ lesser extreme, and $2x =$ common difference. Then the terms will be represented by $y - 4x$, $y - 2x$, y, $y + 2x$, $y + 4x$.

3. Find 4 numbers in arithmetical progression, whereof the product of the extremes is 700, and the product of the means 750.

Ans. 20, 25, 30, 35.

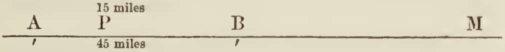
4. The sum of 3 numbers in arithmetical progression is 18, and the sum of their squares 116. What are the numbers? *Ans.* 4, 6, 8.

Let y represent the mean, and x the common difference, then will $y-x$, y and $y+x$, represent the three terms of the progression.

5. A and B are separated by a distance of 45 miles. A travels towards B at the rate of 1 mile the first hour, 2 miles the second hour, 3 miles the third hour, &c. B travels towards A at the rate of 10 miles the first hour, 8 miles the second hour, 6 miles the third hour, &c. When will they come together?

Ans. At the end of the fifth hour, and also at the end of the eighteenth hour.

The second value needs explanation. The formula $l=a-(n-1)d$. which gives the distance travelled by B during the n^{th} hour, shows that B does not travel at all the sixth hour. And since, when $n>6$, l is negative, we conclude that B turns back, and travels in the contrary direction after the sixth hour.



At the end of the fifth hour the two travellers were together at P, 15 miles from A. In 13 more hours the traveller A will be 156 miles from P, at a point M. The traveller B rests during the sixth hour, and at the beginning of the seventh hour turns back in pursuit of A, and at the end of the eighteenth hour from the time of his first starting from B, he will also be at M, 156 miles from B. These results are easily deduced from the formulæ.

When the travellers come together at M, A will have travelled 17 miles, and B 201 miles.

The problem explains most satisfactorily the meaning of a negative solution, and shows that, when distance is the thing to be determined, the negative sign always implies a change of direction.

6. A fugitive from justice has one hour the start of the officers of the law, and travels uniformly at the rate of $7\frac{5}{9}$ miles per hour. The officers travel 5 miles the first hour, 6 the second, 7 the third, &c. How

long will it be from the time of the fugitive's escape until his arrest again? *Ans.* 9 hours, $n = 8$.

7. The mean of seven terms in arithmetical progression is 6, and the product of the extremes 27. What is the common difference, and what the series? *Ans.* $d = 1$. Series, 3, 4, 5, 6, 7, 8, 9.

8. A liquor-seller makes 24 gallons of a mixture of water, brandy, and rum. The number of gallons of the three fluids constitute a progression, of which the number of gallons of water is the least extreme, and that of rum the greatest extreme. The sum of the gallons of water and brandy is equal to the number of gallons of rum. Required the quantity of each. *Ans.* 4, 8, and 12.

9. A man sold a horse upon condition that he should receive 1 cent for the first nail in his shoes, 11 cents for the second nail, 21 cents for the third nail, &c., for the 32 nails in his shoes. How much did he get for him? *Ans.* \$49 and 92 cents.

10. One hundred stones are placed on the same straight line, 4 yards apart. How far will a person have to walk, who puts them one by one in a basket placed on the same straight line, 4 yards from the first stone, and 8 yards from the second stone: the person being supposed to start from the basket? *Ans.* 22 miles, and 1680 yards.

11. Same problem as last, except that the basket is placed at the first stone. *Ans.* 22 miles, and 880 yards.

12. Same problem as 10th, except that the carrier is at the other end of the line, at the 100th stone. Required the last term, the common difference, the first term and the entire distance?

Ans. S = 22 miles, 1280 yards, $l = 8$, $a = 792$.

The progression begins with the second term.

GEOMETRICAL PROGRESSION.

437. A Geometrical Progression is a series of terms, any one of which is formed from that which immediately precedes, by multiplying by a constant quantity. When the constant multiplier is greater than

unity, the series is an increasing progression. Thus 1, 2, 4, 8, 16, 32, 64, &c., constitute an increasing progression, the constant multiplier or *ratio of the progression* being 2. And 32, 16, 8, 4, 2, 1, is a decreasing progression, with the ratio $\frac{1}{2}$. The ratio of a decreasing progression is always unity divided by an entire quantity. It is evident that an increasing progression taken in reverse order will be a decreasing progression, and that the ratio of the latter progression will be the reciprocal of the former. The converse of this is also plainly true. Thus b, $\frac{b}{a}$, $\frac{b}{a^2}$, $\frac{b}{a^3}$, &c., is a decreasing progression whose ratio is $\frac{1}{a}$, a, being supposed an entire quantity, and $\frac{b}{a^3}$, $\frac{b}{a^2}$, $\frac{b}{a}$, and b, constitute an increasing progression, whose ratio is a.

It is plain that a Geometrical Progression differs from a Geometrical Proportion only in its number of terms.

FORMULA FOR THE *Nth* TERM.

438. Let the first term be a, and r the ratio of the progression: then ar will be the second term, ar^2 the third, ar^3 the fourth, and so on. That is, each term is equal to the first term, multiplied by the ratio raised to a power denoted by the number of terms, which precede the required term. Hence, if l represent the n^{th} term, we must have $l = ar^{n-1}$, *that is, the* n^{th} *term is equal to the first term, multiplied by the ratio raised to the* $(n-1)$ *power.* l may represent the last term, in which case $n - 1$ will be the number of terms less one. The formula, $l = ar^{n-1}$, will enable us to determine any term, when the number of preceding terms, the ratio and the first term are known. When $n = 1$, we have $l = a$; when $n = 2$, $l = ar$, &c. When $a = o$, l also $= o$; when $r = o$, l also $= o$; when $a = r$, $l = r^n$.

EXAMPLES.

1. Find the last term of the progression, 1, 2, 4, &c., carried on to 11 terms inclusive. *Ans.* $l = 1024$.

2. Find the n^{th} term of the progression, 3, 9, 27, &c. *Ans.* 3^n.

3. Find the 21st term of the progression, 2, 8, 32, 128, &c. *Ans.* 2199023255552.

4. Find the 6th term of the series, 64, 16, 4, &c. *Ans.* $\frac{1}{16}$.

The n^{th} term being made up of a product and a power, is much greater in geometrical than in arithmetical progression, when r is > 1, and much less in the former than in the latter, when $r < 1$. Thus, a corresponding example to 3, in arithmetical progression, with the first term 2 and common difference 4, would give 82 for the 21st term; and the answer to a corresponding problem to 4, in arithmetical progression, would be 44.

FORMULA FOR THE RATIO OF THE PROGRESSION.

439. The equation, $l = ar^{n-1}$, gives $r = \sqrt[n-1]{\frac{l}{a}}$, that is, *the ratio is equal to the* (n — 1)th *root of the quotient arising from dividing the* nth *term by the* 1st *term.*

This formula can be used when n, l, and a, are known. When $l = a$, we have $r = 1$; when $n = 1$, $r = \infty$; when $n = 2$, $r = \frac{l}{a}$.

EXAMPLES.

1. The 3d term of a geometrical series is 256, and the first term 4. What is the ratio? *Ans.* $r = 8$.

2. The fourth term of a geometrical series is 2401, and the first term 7. Required the ratio. *Ans.* $r = 7$.

3. The first term of a geometrical series is 50, and the fourth term 6250. Required the ratio. *Ans.* $r = 5$.

4. The first term of a geometrical series is 6250, and the fourth term 50. Required the ratio. *Ans.* $r = \frac{1}{5}$.

5. The first term of a geometrical series is a, and the $(m + 1)^{th}$ term a^2. Required the ratio. *Ans.* $\sqrt[m]{a}$.

Corollary.

440. The formula, $r = \sqrt[n-1]{\frac{l}{a}}$, enables us to insert any number of means between two extremes. If we wish to introduce m means between a and l, then the total number of terms will be $m + 2$. Hence,

$n-1=m+2-1=m+1$, and $r=\sqrt[m+1]{\frac{l}{a}}$. Thus, let it be required to insert 3 means between 2 and 32; then, $r=\sqrt[4]{\frac{32}{2}}=\sqrt[4]{16}=2$. And the series is 2, 4, 8, 16, 32.

EXAMPLES.

1. Find three means between $\frac{1}{12}$ and 1728, and the ratio of the progression. *Ans.* 1 . 12 . 144, and $r=12$.

2. Find two means between $\frac{1}{3}$ and 576000.

Ans. 40 and 4800.

FORMULA FOR THE SUM OF THE TERMS.

441. If we multiply a series of terms in Geometrical Progression by the ratio, a new series will be produced, whose first term will be the same as the second of the old series, and whose other terms will all be the same as the corresponding terms of the original series, except its last term. It is plain, then, that if the old series be subtracted from the new, all the terms will be cancelled except the first of the old series and the last of the new.

Take, for example, the series,	3, 9, 27, 81, 243.
And multiply each term by the ratio,	9, 27, 81, 243, 729.
Subtracting the first from the	3, 729.

second series, we have left but two terms.

Let S represent the sum of n terms in geometrical progression. Then, $S=a+ar+ar^2+ar^3 \ldots\ldots ar^{n-1}$.
Multiply by r, and we have $Sr=ar+ar^2+ar^3 \ldots\ldots ar^{n-1}+ar^n$.
Hence, $Sr-S=ar^n-a$, and $S=\frac{a(r^n-1)}{r-1}$.

That is, *the sum of the terms is equal to the first term into the difference between unity and the ratio raised to a power denoted by the number of terms, and this product divided by the ratio, less* 1. This formula can be used when the first term, the ratio, and number of terms are known, or can be determined. Since $l=ar^{n-1}$, then, $lr=ar^n$, and the formula may be written $S=\frac{lr-a}{r-1}$; that is, *the* *sum of the terms is equal to the last, or* n^{th}, *term into the ratio, minus the first term, and this difference divided by the ratio, less* 1.

When $n = 0$, the second members of the equations in S and Sr will cancel each other, and give $S = 0$; but when $r = 1$, the equation in S becomes $S = na$, whilst the formula, $S = \frac{a(r^n - 1)}{r - 1}$, becomes $S = \frac{0}{0}$. This is generally the symbol of indetermination, but, in the present instance, indicates a vanishing fraction. The common factor to the two terms of the fraction is plainly $r - 1$, because $r - 1$ is an exact divisor of the numerator. Performing the division, we have $S = \frac{a\,(r^n - 1)}{r - 1}$ $= a\,(r^{n-1} + r^{n-2} + r^{n-3} + \ldots\ldots r + 1)$. Now, making $r = 1$, we have $S = na$

442. When the progression is decreasing, Sr is less than S; hence, to find S, we must take Sr from S. Then, $S - Sr = a - ar^n$, or $S = \frac{a\,(1 - r^n)}{1 - r} = \frac{a \quad 1 - lr}{1 - r}$, that is, *the sum of the terms of a decreasing progression is equal to the first term into unity, minus the last, or* n^{th}, *term into the ratio, and this difference, divided by one, minus the ratio.*

EXAMPLES.

1. The first term is 2, the ratio 2, and the number of terms 9. What is the sum of the terms? *Ans.* 1022.

2. The first term is 512, the last term 2. Required the sum of the 9 first terms of the progression. *Ans.* 1022.

3. Required the sum of the series, $1 + x + x^2 + x^3 \ldots\ldots\ldots x^{m-1}$.

Ans. $\frac{x^m - 1}{x - 1}$.

4. Required the sum of the series, $x^{m-1} + x^{m-2} + \ldots\ldots x + 1$.

Ans. $\frac{1 - x^m}{1 - x}$.

5. Required the sum of the series, $x^{m-1} + x^{m-2}y + x^{m-3}y^2 \ldots\ldots + y^{m-1}$.

Ans. $\frac{y^m - x^m}{y - x}$, or $\frac{x^m - y^m}{x - y}$.

6. Required the sum of the series, $x^2 + ax + a^2$.

Ans. $\frac{a^3 - x^3}{a - x}$, or $\frac{x^3 - a^3}{x - a}$.

443. The two equations, $l = ar^{n-1}$, and $s = a\frac{(r^n - 1)}{r - 1}$, contain five quantities, any two of which can be determined when the other three are known. Thus, when a, l and n, are known, we obtain, by combining these equations, and eliminating r, $s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.

In like manner, by eliminating l, we get $a = \frac{(r-1)s}{r^n - 1}$; and, by eliminating a, we find $l = \frac{(r^n - r^{n-1})s}{r^n - 1}$.

AN INFINITE DECREASING PROGRESSION.

444. The formula for the sum of the terms of a decreasing progression takes a remarkable form when the series is infinite.

For, $S = \frac{a - r^n}{1 - r}$ may be written, $S = \frac{a}{1-r} - \frac{r^n}{1-r}$.

Now, since r is a fraction whose numerator is unity, like the fraction $\frac{1}{5}$, it is plain that, when $n = \infty$, r^n will be zero. Hence, $\frac{r^n}{1-r}$ will be zero, and that part of the formula may be neglected when the number of the terms is infinite, and we may write $S = \frac{a}{1-r}$; that is, *the sum of the terms of an infinite decreasing progression is equal to the first term divided by unity, minus the ratio of progressions.*

This formula will give the *limit* of the series, that is, a value which the sum of the terms cannot exceed. Thus, the decimal ·33333, &c., may approach infinitely near to $\frac{1}{3}$, but can never exceed it.

EXAMPLES.

1. Find the limit of the value of the decimal, ·666666, &c.

Ans. $\frac{2}{3}$

For $a = \frac{6}{10}$, and $r = \frac{1}{10}$; hence, $S = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{9} = \frac{2}{3}$.

2. Find the limit of the value of the decimal, ·1111111, &c.

Ans. $\frac{1}{9}$.

3. Find the limit of the value of the decimal, ·55555, &c.

Ans. $\frac{5}{9}$.

4. Find the limit of the value of ·99999, &c. *Ans.* 1.

5. Find the limit of the value of ·88888, &c. *Ans.* $\frac{8}{9}$.

6. Find the limit of the value of the series, $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \text{\&c.}$
Ans. $1+\frac{1}{3}$.

7. Find the limit of the value of $\frac{1}{4} + \frac{1}{16} + \frac{1}{64}$, &c. *Ans.* $\frac{1}{3}$.

8. Find the limit of the value of the series, $20 + 4 + \frac{4}{5} + \frac{4}{25} + \text{\&c.}$
Ans. 25.

9. Find the limit of the value of $\frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \text{\&c.}$
Ans. 1.

10. How much will the sum of the series, $\frac{4}{5} + \frac{4}{25}$, &c., differ from unity, when four terms only are taken? *Ans.* By $\frac{1}{625}$

Use the expression, $\frac{ar^n}{1-r}$, which can only be neglected when $n = \infty$.

11. Find the sum of the series, $42 + 6 + \frac{6}{7} + \frac{6}{49} + \frac{6}{343}$, &c.
Ans. 49.

12. Find the limit of the value of $6 + \frac{6}{7} + \frac{6}{49} + \frac{6}{343} + \text{\&c.}$
Ans. 7.

13. By how much will the sum of the series, $6 + \frac{6}{7} + \frac{6}{49} + \frac{6}{343} +$ &c., differ from 7, when three terms only are taken.
Ans. By $\frac{1}{49}$.

GENERAL EXAMPLES.

1. The first term of a geometric progression is 4, the last term 32, and the number of terms 4. Required the sum of the terms.
Ans. 60.

2. There are three quantities in geometric progression, their sum is a, and the sum of their squares, b^2. What are the quantities?

Ans. $y = \frac{a^2 - b^2}{2a}$, $x = \frac{a^2 + b^2}{4a} + \frac{\sqrt{10a^2b^2 - 3b^4 - 3a^4}}{4a}$, $z = \frac{a^2 + b^2}{4a} - \frac{\sqrt{10a^2b^2 - 3b^4 - 3a^4}}{4a}$.

For, let x, y and z, denote the three quantities. Then, $x + y + z = a, y^2 = xz$, and $x^2 + y^2 + z^2 = b^2$. Transposing y to the second member, and squaring the first equation, we get $x^2 + 2xz + z^2 = a^2 - 2ay + y^2$, which, combined with the second, gives $x^2 + y^2 + z^2 = a^2 -$

$2ay$. Subtract the third equation from this, and solve with reference to y; we will find then, $y = \frac{a^2 - b^2}{2a}$. The third equation gives $x^2 + z^2 = b^2 - y^2$. Subtract from this the equation, $2xz = 2y^2$. Then, $x^2 - 2xz + z^2 = b^2 - 3y^2$, and, taking the square root of both members, we find, $x - z = \sqrt{b^2 - 3y^2}$ (M); but, from the first equation, $x + z = a - y$ (N). Adding and substracting (M) from (N), we get $x = \frac{a - y + \sqrt{b^2 - 3y^2}}{2} = \frac{a^2 + b^2}{4a} + \frac{\sqrt{10a^2b^2 - 3b^4 - 3a^4}}{4a}$, and $z = \frac{a - y - \sqrt{b^2 - 3y^2}}{2} = \frac{a^2 + b^2}{4a} - \frac{\sqrt{10a^2b^2 - 3b^4 - 3a^4}}{4a}$.

The problem might have been solved by the ordinary method of elimination, but it would have led to an equation of the fourth degree.

3. Two travellers were separated by a distance of 36 miles. The one in advance travelled 1 mile the first day, 5 miles the second day, 9 the third day, and so on. The one in the rear travelled 1 mile the first day, and increasing his rate in a geometrical ratio, travelled 64 miles on the seventh day, when he overtook the advanced traveller. Required the uniform rate of increase of the second traveller's daily distance. *Ans.* $r = 2$ miles.

4. Find r when s, a and l are known. *Ans.* $r = \frac{S - a}{S - l}$.

Problem 3 is but a particular case of problem 4.

5. There are four quantities in geometric progression, the sum of the first two is a, and of the last two, b. What is the ratio and what are the numbers?

Ans. $r = \sqrt{\frac{b}{a}}$. Numbers, $\frac{a}{1 + \sqrt{\frac{b}{a}}}$, $\frac{a\sqrt{\frac{b}{a}}}{1 + \sqrt{\frac{b}{a}}}$, $\frac{b}{1 + \sqrt{\frac{b}{a}}}$, and $\frac{b\sqrt{\frac{b}{a}}}{1 + \sqrt{\frac{b}{a}}}$.

6. There are four numbers in geometric progression, the sum of the first two is 30, and of the last two, 750. What is the ratio, and what are the numbers? *Ans.* $r = 5$. Numbers, 5, 25, 125 and 625.

7. There are three quantities in geometric progression, the first is a, and the second b. Required the third. *Ans.* $\frac{b^2}{a}$.

8. There are three numbers in geometric progression, the first is 4, and the second 20. Required the third. *Ans.* 100.

9. Find a, when l, r and S are known.

Ans. $a = S - (S - l)r$.

10. The ratio of a geometric progression is 5, the sum of the terms 780, and the last term 625. Required the first term. *Ans.* 5.

11. Find l, when a, r and S are known.

Ans. $l = S - \frac{S - a}{r}$.

12. A rich, but charitable man, gave $20 to the American Tract Society, twice as much to the Board of Foreign Missions, four times as much to the Board of Domestic Missions, and so on in the same ratio. His last contribution was to his own church, and his entire charity amounted to $1020. How much did he give his church?

Ans. $520.

13. A man sold a horse upon the following terms: He was to receive one cent for the first nail in the horse's shoes, two cents for the second nail, four cents for the third, and so on for the 32 nails in the horse's shoes. Required the price of the horse.

Ans. 4,294,967,295 cents.

14.. A gentleman made a donation to a charitable institution, upon the following conditions: Five hundred dollars were to be received the first year, $250 the second year, $125 the third year, and so on forever. Required the amount of the donation. *Ans.* $1000.

15. A gentleman wishes to bequeath $3000 to a Benevolent Association in the following manner: Two thousand dollars to be given the first year to meet the present wants of the association, and the donation every year after, forever, to be in a decreasing ratio. What must the ratio be? *Ans.* $\frac{1}{3}$.

16. A gentleman invested $1000 for the benefit of a charitable institution, so that one half less should be drawn from it each year than the preceding year, forever. How much must be drawn the first year?

Ans. $500.

INEQUALITIES.

445. An inequality is an expression to signify that two quantities are unequal to each other. Thus, $A > B$, is an inequality indicating that A is greater than B. For equalities, it matters not on which side of the sign of equality the two quantities are written. Thus, $A = B$ may be written $B = A$. But, for inequalities, there can be plainly no transposition of the two quantities without a corresponding inversion of the sign. In the expression, $A > B$, if A be changed to the second member, and B to the first, there must be an inversion of the sign. For, let m be the quantity which, added to B, makes it equal to A, then, $A = B + m$, or $B + m = A$. Then $B < A$. So, B and A have changed places, with a corresponding inversion of sign.

Inequalities are used to determine the limits between which quantities are found, and are of frequent application in the higher mathematics. As a simple illustration, suppose we have the two inequalities, $x > 5$, and $x < 10$. We see that x must be some number between 5 and 10, and these numbers are the *limits* to its values. When the enunciation of the problem restricts the solution to positive quantities, there are but two limits, the superior positive limit, and the inferior positive limit. So, if the solution be restricted to negative quantities, there are but two limits, the superior negative limit, and the inferior negative limit. In general, however, there are four limits, two superior and two inferior.

Two inequalities are said to subsist in the same sense, when the greater quantity lies on the same side of the sign in both of them; and they subsist in a contrary sense, when the greater quantity in one inequality lies on a different side of the sign from that occupied by the greater quantity in the other inequality.

Thus, $4 > 2$, and $5 > 3$, subsist in the same sense; so also, $2 < 4$, and $3 < 5$. But $4 > 2$, and $3 < 5$, subsist in a contrary sense.

When two quantities are negative, that one is the least, algebraically, which contains the greatest number of units. Thus, $-15 < -10$, $-5 > -7$.

446. There are ten important principles, belonging to the subject of inequalities, which require to be demonstrated.

1. If the same quantity be added to, or subtracted from, the two members of an inequality, the resulting inequality will subsist in the same sense.

For, let $A > B$, and m the difference between A and B, then, $A = B + m$. Now, add or subtract the same quantity, c, from both members, and there results $A \pm c = B + m \pm c$. Hence, of course, $A \pm c > B \pm c$, and the inequality subsists in the same sense.

This principle enables us to transpose terms from one member of the inequality to the other. Take, $x + 2 > a$; add -2 to both members, and there results $x > a - 2$. So, transposition is effected in inequalities, as in equations, by a change of sign.

2. If the two members of an inequality are multiplied by a positive quantity, the resulting inequality will subsist in the same sense.

For, let $A > B$, and $A = B + m$. Multiply both members by c, and we get $Ac = Bc + mc$. Hence, $Ac > Bc$. This principle serves to free the fractions, if any, of their denominators.

Take $\frac{x}{2} - \frac{a}{4} > b^2$. Clearing of fractions, there results $2x - a > 4b^2$.

3. If both members of an inequality are divided by a positive quantity, the resulting inequality will subsist in the same sense.

For, let $A > B$, and m the difference between A and B. Then $A = B + m$. Dividing both members by c, there results, $\frac{A}{c} = \frac{B}{c} + \frac{m}{c}$. Hence, $\frac{A}{c} > \frac{B}{c}$.

The last principle serves to clear the unknown quantity of its coefficient when positive. Take, $2x > a + 4b^2$, then, $x > \frac{a + 4b^2}{2}$.

The three foregoing principles serve to solve inequalities, when the coefficient of the unknown quantity can be made positive.

4. If two inequalities, subsisting in the same sense, are added together, member by member, the resulting inequality will subsist in the same sense.

For, let $A > B$, and $C > D$. Let m be the difference between A and B, and n the difference between C and D. Then, $A = B + m$, and $C = D + n$. Adding the two equations, member by member, there results $A + C = B + D + m + n$. Hence, $A + C > B + D$.

5. If an inequality be multiplied by a negative quantity, the resulting inequality will subsist in a contrary sense.

For, let $A > B$, or $A = B + m$. Multiply both members by $-c$, and there results $-Ac = -Bc - m$, or $m - Ac = -Bc$. Hence, Ac being numerically greater than Bc, we have, algebraically, $-Ac < -Bc$.

The foregoing principles enable us to eliminate between inequalities as between equalities, when we have two inequalities involving two unknown quantities whose signs are known, and that of one of them in one inequality contrary to that of the same unknown quantity in the other inequality.

$$\text{Take } x + y > 12,$$
$$x - y > 8.$$

Adding member by member we get $x > 10$, and this substituted in the first equation, gives $y > 2$. Let $x = 10 + m$. Then, from the 2d equation $y < 2 + m$. The value of y is then fixed between the limits of 2 and $2 + m$.

6. If two inequalities, subsisting in the same sense, are subtracted member by member, the resulting inequality may subsist in the same or a contrary sense.

For, let $A > B$, and $C > D$. Then, $A = B + m$, and $C = D + n$. Subtracting member by member, there results $A - C = B + m - (D + n)$; or, $A - C = B - D + m - n$.

Now, if $m > n$, $A - C$ will be equal to $B - D$ increased by a positive quantity, and, of course, will be greater than it. But, if $n > m$, $A - C$ will not be equal to $B - D$, until it has been diminished by the difference between n and m.

In that case, $A - C < B - D$.

Take, $8 > 4, \quad m = 4,$
$6 > 3, \quad n = 3.$
$8 - 6 > 1.$

The resulting equality subsists in the same sense, since $m > n$.

But, take $8 > 6, \quad m = 2,$
$6 > 1, \quad n = 5.$
$8 - 6 < 5.$

The resulting inequality subsists in a contrary sense, since $n > m$.

8. If both members of an inequality are essentially positive, they may be squared without altering the sense of the inequality.

For, let $A > B$, or $A = B + m$, then $A^2 = B^2 + 2Bm + m^2$. Hence, $A^2 > B^2$.

9. If the two members of an inequality have contrary signs, the resulting inequality, after squaring, may subsist in the same or a contrary sense.

For, let $A > -B$, or $A = -B + m$. Then, $A^2 = B^2 - 2Bm + m^2$. If $m^2 > 2Bm$, then, $A^2 > B^2$; for, A^2 is equal to B^2, increased by the difference between m^2 and $2Bm$. But, if $m^2 < 2Bm$, then, $A^2 < B^2$ by the difference between $2Bm$ and m^2.

Take, $3 > -2$; then, $m = 5$, $m^2 = 25$, and $2Bm = 20$.
Squaring, we have, $9 > 4$, since $m^2 > 2Bm$.
But take, $3 > -6$; then $m = 9$, $m^2 = 81$, and $2Bm = 108$.
Squaring, we get, $9 < 36$, an inverted sign, since $m^2 < 2Bm$.

When, therefore, the sign of either member is unknown, it is not permitted to square the inequality.

10. When the two members of an inequality are divided by a negative quantity, the resulting inequality will subsist in a contrary sense.

For, let $A > B$, or $A = B + m$. Dividing both members by $-c$, there results $-\frac{A}{c} = -\frac{B}{c} - \frac{m}{c}$. Since $-\frac{A}{c}$ is numerically greater than $-\frac{B}{c}$, it must be algebraically less.

For equations of high degrees there are generally four limits: a superior positive and a superior negative limit, and an inferior positive and an inferior negative limit. But the conditions of the problem may restrict the solutions to two limits, and even to one limit.

A single equation of the first degree can have but one limit, this may be superior or inferior, positive or negative. Limits are determined by means of inequalities.

EXAMPLES.

1. The value of x in an equation is such that, twice the value increased by unity cannot be less than 7. What is the lower limit of the value? We have, from the conditions, $2x + 1 > 7$. Hence, $x > 3$. The inferior positive limit is then 3, and any number above 3 may be the value required.

2. The value of x in a simple equation of the first degree is known to be such, that twice the value, plus unity, is greater than 7, and that

three times the value, diminished by 4, is less than 11. Between what limits does the value lie?

Ans. Superior positive limit, 5; inferior positive limit, 3.

If the value be known to be a whole number, 4 must be that number.

3. A person desirous of giving some cents to a certain number of beggars, found on examination that, to give them three cents apiece, would require more than double of what he had about him, and that, to give them 2 cents apiece, would require more than the difference between all he had and 35 cents. How many beggars were there, and how many cents had the person?

From the conditions, $3x > 2y$,

and, $$2x > 35 - y.$$

Multiplying the second inequality by 2, and adding member by member, there results, $7x > 70$, or $x > 10$. This value substituted in the first inequality, gives $y < 15 + \frac{3}{2}m$; m representing the excess of x over 10. The value of x, substituted in the second inequality, gives $y > 35 - 20 - 2m$; or $y > 15 - 2m$. Suppose $m = \frac{1}{2}$, then. $y < 15\frac{3}{4}$, and $y > 14$.

4. A person desiring to give some money to 11 beggars, found that, to give them 3 cents apiece, would require more than twice as much money as he had about him, and that, to give them 2 cents apiece, would require more than the difference between 37 cents and the number of cents about him. Required the number of cents he had.

Ans. $x < 16\frac{1}{2}$, and $x > 15$; hence, $x = 16$.

The nature of the problem makes the solution exact in this case, but it is very seldom so.

5. Find the negative limits of x in the inequalities,

$$x + 3 < -5,$$
$$x - 3 > -7.$$

Ans. $x < -8$, and $x > -4$.

6. Find the limit of the value of x in the inequality, $ax + b + c > d - fx$.

Ans. Inferior limit, $\dfrac{d - b - c}{a + f}$

GENERAL THEORY OF EQUATIONS.

447. *The general theory of equations* has for its object, the investigation of properties common to equations of every degree and of every form.

We will confine ourselves mainly to the examination of equations involving but one unknown quantity.

The most general form of an equation of the m^{th} degree with one unknown quantity, is $x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots\ldots + Tx + U = 0$.

The coefficient of the first term is plus unity, and the other coefficients, positive or negative, entire or fractional, rational or irrational.

A value has been defined to be that which, substituted for the unknown quantity, will make the two members equal to each other. Since, in the general equation of the m^{th} degree, the second member is zero, a value substituted for x, must reduce the first member to zero also. Hence, in our discussion, we may define a value to be that which, substituted for the unknown quantity in the equation, will reduce the first member to zero.

GENERAL PROPERTIES OF EQUATIONS.

First Property.

448. If any quantity, a, be a value of x in the equation, $x^m + Px^{m-1} + \ldots Tx + U = 0$, the first member of this equation will be exactly divisible by $x - a$.

For, let Q' be the quotient resulting from the division of the first member by $x - a$, and let R' be the remainder, if any, after division. We shall then have the identical equation, $x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = Q'(x - a) + R'$. But, for $x = a$, the first member is, by hypothesis, equal to zero, and the second member reduces to R'. Hence, we have $0 = 0 + R'$, and, therefore, $R' = 0$, and the division is exact.

Second Property.

449. If the first member of an equation, of the form of $x^m + Px^{m-1} + Qx^{m-2} + \ldots Tx + U = 0$, be divisible by $x - a$, then a will be a value in this equation.

For, calling Q′ the quotient, we will have $x^m + Px^{m-1} + Qx^{m-2} + \dots Tx + U = Q'(x - a)$. Hence, then, $Q'(x - a) = 0$. But, when the product of two factors is equal to zero, the equation can be satisfied by placing either factor equal to zero. We have then a right to place $x - a = 0$, from which there results $x = a$. Therefore, a is a value in the equation, $Q'(x - a) = 0$, and, consequently, in the given equation.

Corollary.

1st. It follows that, in order to ascertain whether any polynomial is divisible by $x - a$, we have only to substitute a for x, wherever x occurs, and see whether the polynomial reduces to zero.

2d. Hence, also, if a polynomial is divisible by $x - a$, a will be a value in the equation formed by placing the polynomial equal to zero.

3d. We may also diminish the degree of an equation, when we know one or more of its values, by dividing its first member successively by the binomial factors corresponding to these values. The division by each binomial factor will reduce the degree of the equation by unity.

4th. Numerical equations can frequently be solved by means of the first two properties. Literal equations can also be solved in the same way, but more rarely: we have only to ascertain what number or quantity will satisfy the equation; then, by dividing out the binomial factor corresponding to this value, we will have a new equation of a degree lower by unity. A second value may be found from this equation, and the factor corresponding to it divided out. Thus, we may continue the process until the given equation is reduced to one of the second degree, which can be solved by known rules.

EXAMPLES.

1. Solve the equation, $x^3 - 6x^2 + 11x - 6 = 0$.

We see that $x = 1$ will satisfy this equation. Hence, $x - 1$ is a divisor. Dividing by $x - 1$, we get a quotient, $x^2 - 5x + 6 = 0$, by solving which we get $x = 2$ and 3. Hence, the three values are 1, 2, and 3.

2. Solve the equation, $x^4 - 5x^2 + 4 = 0$.

We find $+1$ to be a value, and, dividing by $x - 1$, we get a new equation, $x^3 + x^2 - 4x - 4 =$, which is satisfied for $x = -1$. Dividing by $x + 1$, we get a new equation, $x^2 - 4 = 0$, which gives the two values, $x = +2$, and -2. The four values are then, $+1$, -1, $+2$, and -2.

3. Solve the equation, $x^5 + 3x^4 - 5x^3 + 4x + 12 - 15x^2 = 0$.
Ans. $+1, -1, +2, -2$, and -3.

4. Solve the equation, $x^4 - a^2x^2 - b^2x^2 + a^2b^2 = 0$.
Ans. $+a, -a, +b, -b$.

5. Solve the equation, $x^5 - a^2x^3 - b^2x^3 + a^2x^2 + b^2x^2 - x^4 + a^2b^2x - a^2b^2 = 0$. *Ans.* $+a, -a, +b, -b$, and $+1$.

6. Solve the equation, $x^3 + x^2 - 4x - 4 = 0$.
Ans. $+2, -2, -1$.

Third Property.

449. Every equation, with one unknown quantity, has as many values for this unknown quantity as is denoted by the degree of the equation, and has no more.

Let us assume the equation, $x^m + Px^{m-1} + Qx^{m-2} + \ldots Tx + U = 0$, which is of the m^{th} degree. It is to be shown that it has m values and no more. We will also assume that every equation has at least *one value*. Let a be one value, then $x - a$ is a divisor. Let $x^{m-1} + P'x^{m-2} + Q'x^{m-3} + \ldots\ldots T'x + U'$, represent the quotient of the division of the first member by $x - a$. Then the given equation will assume the form, $(x - a)\ (x^{m-1} + P'x^{m-2} + Q'x^{m-3} + \text{\&c.}) = 0$. (A), in which, the coefficients, P', Q', are different from the original coefficients, P, Q, &c. Now, since, in equation (A), we have the product of two factors equal to zero, we have a right to place either equal to zero, let us place $x^{m-1} + P'x^{m-2} + Q'x^{m-3} + \text{\&c.} = 0$ (B). Equation (B) will also have one value; suppose it negative and equal to $-b$. Then, by the first property, $x - (-b) = x + b$ will be a divisor. The first member of (B) can then be decomposed into two factors, one of which will be $x + b$, and the other the quotient arising from the division of the first member by $x + b$. Hence, (B) becomes $(x + b)\ (x^{m-2} + P''x^{m-3} + Q''u^{m-4} + \text{\&c.}) = 0$ (C). And (A) can be put under the form of $(x - a)\ (x + b)\ (x^{m-2} + P''x^{m-3} + Q''x^{m-4} + \text{\&c}) = 0$. It is plain that, by putting the second factor of (C) equal to zero, we will get another value, and, consequently, another divisor. By continuing this process, the degree of x in the successive quotients will be diminished by unity each time, and, after $m-1$ divisions, we will obtain a quotient of the first degree in x, from which, of course, one value of x will be found. We supposed the first value positive, and the second negative, let us attribute the double sign, $\pm$, to the remaining values, c, d, e, &c., since they may be either positive or negative. The first member of the given equation can then be decomposed into m binomial factors of the first degree in x, and we will have, $x^m + Px^{m-1} + Qx^{m-2} + \ldots\ldots + Tx + U = (x-a)\ (x+b)\ (x \mp c)\ (x \mp d)$ &c. $= 0$.

But, since each factor corresponds to a value, and since there are m *divisors*, or factors, there must be m values.

Scholium.

1st. Had we known that the first member of an equation of the m^{th} degree could be decomposed into m binomial factors of the first degree with respect to x, we could readily have shown that the equation must contain m values. For, it can be satisfied in m ways, by placing each of its m factors equal to zero, and each factor, so placed, will give a value. Moreover, since the equation can be satisfied only in m ways, it contains no more than m values.

2d. It does not follow that all the values must be different. Any number of them, even all of them, may be equal. Thus, the equation, $x^2 - 2x + 1 = 0$, contains two values, each equal to $+1$. The equation, $(x-a)^m = 0$, contains m values, each equal to $+a$ The equation, $(x-a)^m(x+b)^n = 0$, contains m values, each equal to $+a$, and n values, each equal to $-b$. And so for other equations.

3d. An equation of the third degree, such as $(x-a)\ (x-b)\ (x-c) = 0$, will contain three divisors of the first degree, $(x-a)$, $(x-b)$, and $(x-c)$; three of the second degree, $(x-a)\ (x-b)$, $(x-a)\ (x-c)$, and $(x-b)\ (x-c)$; and one of the third, $(x-a)\ (x-b)\ (x-c)$. These divisors are evidently equal to the number of combinations which can be formed by combining 3 letters in sets of 1 and 1, 2 and 2, 3 and 3. So, likewise, if we take the m factors of an equation and multiply them two and two, three and three, &c., we shall evidently obtain as many divisors of the second degree as we can form combinations of m letters, taken two and two; and as many divisors of the third degree as there are combinations of m letters taken three and three,

&c. The given equation will then have m divisors of the first degree, in x, $\frac{m(m-1)}{1 \cdot 2}$ divisors of the second degree, $\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}$ of the third degree, &c.

Fourth Property.

451. All the coefficients, after the first, of an equation of the m^{th} degree, are functions of the values.

Suppose the general equation of the m^{th} degree, $x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0$, contains the m values, $\pm a$, $\pm b$, $\pm c$, $\pm d$, &c. The equation can then be put under the form of $(x \mp a)$ $(x \mp b)$ $(x \mp c)$ $(x \mp d)$, &c., to m factors $= 0$. By actually performing the indicated multiplication, we will get,

$x^m \mp a$	$x^{m-1} \ddagger ab$	$x^{m-2} \mp abc$	$x^{m-3} \ldots\ldots \mp abcd$, &c. $= 0$.
$\mp b$	$\ddagger ac$	$\mp abd$	
$\mp c$	$\ddagger ad$	$\mp acd$	
$\mp d$	$\ddagger$ &c.	$\mp$ &c.	
&c.	$\ddagger bc$	$\mp bcd$	
	&c.	$\mp$ &c.	

The upper row of signs belonging to the positive value, and the lower to the negative.

We observe the following relation between the coefficients and the values:

1. The coefficient of the second term is the sum of all the values of the unknown quantity, with their signs changed.

2. The coefficient of the third term is the sum of the products of all the values, taken two and two, with their respective signs.

3. The coefficient of the fourth term is the sum of the products of all the values, taken three and three, with their signs changed.

4. The last term is the continued product of all the values of the unknown quantity, with their respective signs if the degree of the equation be even, or with their signs changed if it be odd.

We have supposed in the preceding demonstration that all the values were positive or all negative. But the law of formation for the coefficients is evidently the same when some of the values are positive and some negative.

Thus, let the values be $+a$, $+b$, and $-c$.

Then, $(x-a)(x-b)(x+c)=0$, or

$$x^3 \begin{array}{l|l|l} -a & x^2 + ab & x + abc = 0. \\ -b & \quad -ac & \\ +c & \quad -bc & \end{array}$$

And we see that coefficients are formed in accordance with the above law.

452. The preceding properties show several important things in relation to the composition of an equation.

1. If the coefficient of the second term of an equation be zero, and, consequently, the second term be wanting, the sum of the positive values must be equal to the sum of the negative values.

2. If the signs of the terms of an equation be all positive, the values must be all negative. For, an equation cannot have all its terms positive unless its binomial factors are of the form, $(x+a)(x+b)(x+c)$, &c.; which factors, placed equal to zero, will give the negative values, $-a$, $-b$, $-c$, &c.

3. If the signs of the terms of an equation be alternately $+$ and $-$, the values of the unknown quantity will be all positive. For, in this case, the first member of the equation must be made up of the factors, $(x-a)(x-b)(x-c)$, &c., corresponding to positive values.

4. Since the last term, irrespective of its sign, is the continued product of the values, it follows that, when the last term is zero, one value, at least, must be zero.

5. It follows, also, from the composition of the last term, that, in seeking for a value, we need only seek among the divisors of the last term, since every value must be a divisor of that term.

6. If we know one value, the coefficient of the second term will give the sum of all the rest, and the last term, divided by this value, with or without its sign changed, will give the product of all the rest.

Corollary.

453. 1st. The last two consequences enable us to solve equations with facility when all the values except two are known.

Take, as an example, the equation, $x^3 - \frac{3}{4}x^2 - \frac{3}{8}x + \frac{1}{8} = 0$.

The values must be sought among the divisors of the last term: $+1$ is a divisor of the last term, and *may*, therefore, be a value; upon trial, we find that it will satisfy the equation, and *is*, therefore, a value. Calling the other two values x and y, we will have, $-(1+x+y) = -\frac{3}{4}$, the coefficient of the second term. Hence, $x+y=-\frac{1}{4}$. We have, also (sixth consequence), $\frac{xy}{1} = -\frac{1}{8}$. Combining these two equations, we get $x(-\frac{1}{4}-x) = -\frac{1}{8}$, or $x^2+\frac{1}{4}x = +\frac{1}{8}$. And the values of x and y are found to be $+\frac{1}{4}$, and $-\frac{1}{2}$. The equation has, of course, but three values, since it is an equation of the third degree, (Art. 450).

2d. When all the values are known except two, we can tell whether these values are real or imaginary without actually finding them.

Thus, take the equation, $x^4-6x^2-16x+21=0$.

The exact divisors of the last term are $+1$, $+3$, and $+7$, -1, -3, and -7; the values must be sought among these numbers. We find, upon trial, that $+1$ and $+3$ are values. The second term of the equation being wanting, the sum of the other two values must be -4, and their product $\frac{21}{1.3}=7$. But 4 is the greatest product which can be given by two numbers whose sum is 4. Hence, the other two values of the equation are imaginary. And, in fact, by pursuing the preceding process, we find them to be $-2+\sqrt{-3}$, and $-2-\sqrt{-3}$

EXAMPLES.

1. Solve the equation, $x^3-2ax^2-9a^2x+18a^3=0$.

   Ans. $+2a$ $+3a$, and $-3a$.

2. Solve the equation, $x^3-4x^2+x-4=0$.

   Ans. $+4$, $+\sqrt{-1}$, and $-\sqrt{-1}$.

3. Solve the equation, $x^4-2x^2-8=0$.

   Ans. $+2$, -2, $+\sqrt{-2}$, $-\sqrt{-2}$.

454. 3d. If the values are known, the equations which give those values can be formed in two ways, either by multiplying together the binomial factors corresponding to those values (Art. 450), or by forming the coefficients, (Art. 451).

EXAMPLES.

1. Form the equation whose values are $+1$, -1 and -2.
Ans. $x^3 + 2x^2 - x - 2 = 0$.

For the factors, multiplied together, and placed equal to zero, give

$$(x-1)(x+1)(x+2) = 0 \quad \text{(A) or } x^3 + 2x^2 - x - 2 = 0.$$

We see that the three factors of (A), placed separately equal to zero, will give the values, $+1$, -1, and -2, or the problem may be solved by Art. 450.

Coefficient of 2d term $= -(+1-1-2) = +2$
" 3d term $= (+1)(-1)+(+1)(-2)+(-1)(-2) = -1$
" 4th term $= -(+1)(-1)(-2) \ldots\ldots\ldots = -2$.

2. Form, by both methods, the equation whose values are $+4$, $+\sqrt{-1}$, and $-\sqrt{-1}$.
Ans. $x^3 - 4x^2 + x - 4 = 0$.

3. Form, by both methods, the equations whose values are $+2$, -2, $+\sqrt{-2}$, $-\sqrt{-2}$.
Ans. $x^4 - 2x^2 - 8 = 0$.

4. Form, by both methods, the equation whose values are -1, -2, -3, and -4.
Ans. $x^4 + 10x^3 + 35x^2 + 50x + 24 = 0$.

Fifth Property.

455. Every equation may be transformed into another, in which the values of the unknown quantity shall be equal to those of the proposed equation, increased or diminished by a certain quantity.

Let the given equation be

$$x^m + Px^{m-1} + Qx^{m2} + \ldots\ldots + Tx + U = 0,$$

and let it be proposed to transform it into another in y, so that $y = x \pm a$. The principle of transformation is, of course, the same when $y = x + a$, or $= x - a$; we will then confine our discussion to the case, when $y = x - a$. From this equation there results, $x = y + a$. Substituting this value in the proposed equation, it becomes $(y+a)^m + P(y+a)^{m-1} + Q(y+a)^{m-2} + \ldots\ldots\ldots T(y+a) + U = 0$. Developing the different powers of $y + a$ by the binomial formula, and arranging the development according to the descending powers of y, we have the transformed equation,

$$\left.\begin{array}{r|l|l} y^m + ma & y^{m-1} + \dfrac{m(m-1)a^2}{1\,.\,2} & y^{m-2} \;.\;.\;.\;.\; + a^m \\ + P & \qquad + m(m-1)Pa & \qquad\qquad + Pa^{m-1} \\ & \qquad + Q & \qquad\qquad + Qa^{m-2} \\ & & \qquad\qquad + Ta \\ & & \qquad\qquad + U \end{array}\right\} = 0.$$

If the values of y in this equation can be found, the corresponding values of x in the given equation will be equal to these values increased by a.

Corollary.

456. The preceding transformation enables us to reduce the number of terms in an equation. For, since the quantity, a, is entirely arbitrary, such a value may be given to it as will reduce the coefficient of any term in the transformed equation to zero, and consequently make the term itself disappear. Suppose we wish to free the transformed equation of its second term; we have only to place the coefficient of that term equal to zero, and find the value of the quantity, a, that makes it zero. Of course, then, this value attributed to a, will cause the disappearance of the second term of the equation. Placing $ma + P = 0$, we get $a = -\frac{P}{m}$. Then, $x = y - \frac{P}{m}$. Hence, for transforming one equation into another, in which the second term is wanting, the following

RULE.

Substitute for the unknown quantity a new unknown quantity, connected with the quotient arising from dividing the coefficient of the second term of the given equation, with its sign changed by the degree of the equation.

EXAMPLES.

1. Transform the equation, $x^2 - 4x = 5$, into another, in which the second term shall be wanting. *Ans.* $x = y - (-\frac{4}{2}) = y + 2$.

The transformed equation will then be $(y+2)^2 - 4(y+2) = 5$, or $y^2 - 4 = 5$. Hence, $y^2 = 9$, and $y = \pm 3$; and, $x = y + 2 = 5$, or -1; the same result that we would obtain by solving the given equation.

2. Transform the equation, $x^3 - 3x^2 = -2$, into another, in which the second term shall be wanting. *Ans.* $y^3 - 3y = 0$.

The transformed equation can be readily solved, one value of y being zero. Hence, one value of the given equation is 1. The transformed equation, in this case and in many others, is simpler than the given equation, and, therefore, more readily solved. The chief object of the transformation is to simplify the form of the equation.

Scholium.

457. 1. The third term of the transformed equation may be made to disappear by giving to a such a value as will satisfy the equation, $\frac{m(m-1)a^2}{1\,.\,2} + (m-1)Pa + Q = 0$. And, since this equation is of the second degree in a, there are two values for a, and, consequently, the third term can be made to disappear in two ways. In like manner, the fourth term can be made to disappear in three ways, the fifth term in four ways, &c.; and the last, or $(m+1)^{th}$ term, in m ways. By recurrence to the derivations of these coefficients from the values, we see that the above results are true. The last term, for instance, being made up of the product of the m values, can be made to disappear by placing either of the m values equal to zero.

458. 2. It may happen that the value of a, which makes the second term disappear, will also cause the disappearance of the third or some other term at the same time. Let us examine under what circumstances the second and third terms will disappear together. By placing the coefficient of the second term, $Pa + m$, equal to zero, we get $a = -\frac{P}{m}$, and, by placing that of the third term, $\frac{m(m-1)a^2}{1\,.\,2} + (m-1)Pa + Q$, also equal to zero, we get $a = -\frac{P}{m} \pm \sqrt{\frac{P^2}{m} - \frac{2Q}{m(m-1)}}$.

Now, it is plain that the values of a in this equation will be identical with the last, that is, both $-\frac{P}{m}$, when the radical disappears. Placing the radical equal to zero, we get $P^2 = \frac{2mQ}{m-1}$. Whenever, then, the square of the coefficient of the second term is equal to twice the quotient arising from dividing the product of the degree of the equation into the coefficient of the third term by the degree of the equation, less one, the second and third terms will disappear together.

The same truth may be demonstrated more elegantly by the principle of the greatest common divisor.

Dividing the coefficient of the third term by that of the second, we get, after two divisions, a remainder, $mQ + \frac{(m-1)P^2}{2}$. And, it is evident that, when this remainder is placed equal to zero, we will have made manifest the circumstances under which there will be a common factor between the two coefficients. Placing the remainder equal to zero, we get $P^2 = \frac{2mQ}{m-1}$, as before.

EXAMPLES.

1. Make the second and third terms disappear from the equation, $x^3 - 3x^2 + 3x - 28 = 0$, and find one value of x.

Ans. Transformed equation, $y^3 - 27 = 0$, and $x = 4$.

2. Find two values in the equation, $x^4 + 4x^3 + 6x^2 + 4x - 15 = 0$.

Ans. Transformed equation, $y^4 - 16 = 0$, then, $y = \pm 2$, and $x = +1$, or -3.

Sixth Property.

459. Every equation having the coefficient of the first term plus unity, and the other coefficients entire, will have entire numbers only for its rational values.

Let the proposed equation be

$$x^m + Px^{m-1} + Qx^{m-2} \dots\dots + Tx + U = 0.$$

In which, P, Q, &c., are whole numbers.

If x can have a fractional value in this equation, let this value, reduced to its lowest terms, be $\frac{a}{b}$. Substituting this value for x in the given equation, it becomes

$$\frac{a^m}{b^m} + \frac{Pa^{m-1}}{b^{m-1}} + \frac{Qa^{m-2}}{b^{m-2}} \dots\dots + \frac{Ta}{b} + U = 0.$$

Multiplying by b^{m-1}, and transposing, we get

$$\frac{a^m}{b} = -Pa^{m-1} - Qa^{m-2}b \dots\dots - Tab^{m-2} - Ub^{m-1}.$$

But the second member is an entire quantity, since all its terms are entire. Hence, the first member must be entire. But, since a, by hypothesis, is prime with respect to b, a^m must also be prime with respect to b. The supposition of the proposed equation containing a rational fractional value, has then resulted in the absurdity of making an entire quantity equal to an irreducible fraction. We conclude, therefore, that this supposition is wrong, and that the rational values are all entire. The demonstration is restricted to rational values, because the assumed value, $\frac{a}{b}$, is rational in its form.

Corollary.

460. Articles 452, 456 and 459, are used in solving numerical and literal equations by changing their forms.

Let the proposed equation be $y^4 - 4y^3 - 8y + 32 = 0$. We know that all the rational values are entire (Art. 459), and that they must be found among the divisors of the last term (Art. 452). But as there are so many divisors of the last term, it will be more convenient to employ (Art. 456) to transform the equation into another, whose last term admits of fewer divisors. Make $x = y - \left(\frac{-4}{4}\right) = y + 1$. The transformed equation in x, is $x^4 - 6x^2 - 16x + 21 = 0$, and the divisors of the last term, are $+1$, $+3$, $+7$, $+21$, and -1, -3, -7, -21. On trial of these divisors, we find that $+1$ and $+3$, will satisfy the transformed equation, and, consequently, are values. Hence, $x = y + 1 = 2$ and 4, in the given equation. Dividing the first member of the given equation by $(x - 2)$ (Art. 449), it will be reduced to a cubic equation, and again dividing by $(x - 4)$, it will be reduced to a quadratic, which can be solved.

Seventh Property.

461. *Imaginary values enter equations by pairs.*

We are to show that if $a + b\sqrt{-1}$ is a value in the equation, $x^m + Px^{m-1} + Qx^{m-2} \ldots + Tx + U = 0$, $a - b\sqrt{-1}$ will also be a value.

The truth of the proposition is an evident consequence of Art. 451, for U, the last term, is the product of all the values, and it has been assumed real. But it can only be real, when the imaginary values (if any), enter by pairs, and of such a form as to give a real product when multiplied together. Thus, if we have two imaginary values, $m + \sqrt{-n}$ and $\sqrt{-4}$, we must also have two other values, $m - \sqrt{-n}$ and $-\sqrt{-4}$, otherwise, U will not be real. Or, we may demonstrate the property otherwise, thus: by substituting the assumed value, $a + b\sqrt{-1}$, in the given equation, we will have, $(a + b\sqrt{-1})^m + P(a + b\sqrt{-1})^{m-1} + \ldots\ldots T(a + b\sqrt{-1}) + U = 0$. When these terms are expanded, the odd powers of $b\sqrt{-1}$ will be imaginary, and the even, real. Representing by A all the real terms involving a or b, and by $B\sqrt{-1}$ all the imaginary terms, we will have $A + B\sqrt{-1} = 0$. Now, since $a + b\sqrt{-1}$ is, by hypothesis, a value, the last equation must be satisfied, but this can only be so when $A = 0$, and $B\sqrt{-1} = 0$, since imaginary terms cannot be cancelled by real ones. If $a - b\sqrt{-1}$ be substituted in the equation, the expanded results will be precisely the same, except that the odd powers of $b\sqrt{-1}$ will be negative. Hence, we will have $A - B\sqrt{-1} = 0$. But in order that $a + b\sqrt{-1}$ should be a value, we have seen that $A = 0$ and $B = 0$. Hence, the equation, $A - B\sqrt{-1} = 0$, is satisfied, and that being so, $a - b\sqrt{-1}$ must be a value.

The absurdity of the hypothesis of a single imaginary value may be illustrated by an example.

Let us assume that the three values of an equation of the third degree are $+a$, $+1$, and $+\sqrt{-1}$. The equation must then be $(x - a)(x - 1)(x - \sqrt{-1}) = 0$, or $(x - a)(x^2 - (1 - \sqrt{-1})x + \sqrt{-1}) = 0$. Hence, $x - a = 0$, and $x^2 - (1 + \sqrt{-1})x + \sqrt{-1} = 0$. The second equation, when solved, will give $x = \frac{1 + \sqrt{-1}}{2}$

$$\pm\sqrt{-\sqrt{-1} + \frac{1 + 2\sqrt{-1} - 1}{4}} = \frac{1 + \sqrt{-1}}{2} \pm \sqrt{\frac{-2\sqrt{-1}}{2}}.$$

And we see that we do not get back again the two values, $+1$, and $+\sqrt{-1}$.

Remarks.

462. 1. The product of imaginary values is always positive, and, therefore, the absolute term of an equation, whose values are all imaginary, must be positive.

2. If the second term of an equation containing only imaginary values, is wanting, these values will all be of the form, $\pm\sqrt{-b}$.

3. Every equation of an odd degree has at least one real value for x,

and the sign of this value will be contrary to that of the last term of the equation.

4. Every equation of an even degree, whose last term is negative, has at least two real values for x; one positive, and one negative.

EXAMPLES.

1. One value of x is $4+\sqrt{-10}$, what must a second value be?

Ans. $4-\sqrt{-10}$.

2. Two of the values of an equation of the fourth degree are $+a$, and $-b$, and the last term is $-m$. How are the remaining values?

Ans. Real.

Eighth Property.

463. If the real values of an equation, taken in the order of their magnitudes, be a, b, c, d and e; a being $> b$, $b > c$, &c.; then, if a series of quantities, a', b', c', d', &c.; a' taken greater than a, b' between a and b, c' between b and c; be substituted for x in the proposed equation, the results will be alternately positive and negative.

For, since a, b, c, d, &c., are assumed to be values, the proposed equation can be put under the form of $(x-a)(x-b)(x-c)(x-d)$ .... &c. $=0$.

Substituting, for x, the proposed series of quantities, a', b', c', d', &c., we get the following results.

$(a'-a)(a'-b)(a'-c)(a'-d)$ &c, $=+$ result, since all the factors are positive.

$(b'-a)(b'-b)(b'-c)(b'-d)$ &c., $=-$ result, since first factor only is negative.

$(c'-a)(c'-b)(c'-c)(c'-d)$ &c., $=+$ result, since first two factors only are negative.

$(d'-a)(d'-b)(d'-c)(d'-d)$ &c., $=-$ result, since first three factors only are negative

Remarks.

1. We see that, between the quantities, a' and b', which give the first plus result and the first minus result, there is one value, a. And between the quantities, a' and d', which give the first plus result and the second minus result, there are three values, a, b and c. And, as the

same is evidently true for any number of odd values, we conclude that, if two quantities be successively substituted for x in an equation, and give results, with different signs, between these quantities, there will be one, three, five, or some odd numbers of real value.

2. If any quantity, m, and every quantity greater than m, give results all positive, then m is greater than the greatest value of x in the equation.

3. If the results obtained by substituting two quantities have the same sign, then, between these quantities there are two, four, six, or some even number of values, or no value at all. Thus, between a' and c', which give + results, there are two values, a and b.

Scholium.

464. The preceding property may be demonstrated differently, by employing a principle of frequent application in all branches of mathematics, viz., that a quantity changes its sign in passing through zero and infinity. Let the quantity be $x = a$; suppose $x > a$, the expression is positive; it is zero when $x = a$, and negative when $x < a$. Hence, the quantity $x - a$, changes its sign in passing through zero. Again, take $\frac{+M}{x-a}$. This quantity is positive when $x > a$; it is infinite when $x = a$; and negative when $x < a$. Now, in the results of Art. 463, between the first + result and the first — result, the first member of the proposed equation must have passed through zero, and consequently through a value. In like manner, between the first — result, and the second + result, the first member must again have passed through zero, and consequently through a second value. Then two values must have been passed through between the first two + results, &c. By continuing the same course of reasoning, we might readily show that three values must be passed through between the first positive and second negative result; five values between the first positive and third negative result; two values between the first two negative results; four between the first and third negative results, &c., &c.,

Limit of the Values. — Ninth Property.

465. If we substitute for the unknown quantity the greatest coefficient plus unity, the first term of the equation will be greater than the sum of all the other terms, provided we always affect the greatest coefficient with the positive sign.

Let us resume the equation,

$$x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U = 0.$$

It is plain that the most unfavourable case will be when all the coefficients have the same sign, + or —. We will assume all the coefficients after the first to be negative; the given equation will then be of the form, $x^m - Px^{m-1} - Px^{m-2} - Px^{m-3} - Px^{m-4} \dots\dots - P = 0$, or $x^m - P(x^{m-1} - x^{m-2} - x^{m-3} \dots\dots - 1) = 0$.

Disregarding the sign of P for a moment, we wish to ascertain what value x must have, in order that $x^m > P(x^{m-1} + x^{m-2} + x^{m-3} \dots + x + 1)$. Inverting the order of the terms within the parenthesis, we will have a geometrical progression, whereof, 1 is the first term, x^{m-1} the last term, and x the ratio of the progression. The formula, $S' = \frac{lr - a}{r - 1}$, will then give the value of the quantity within the parenthesis, $= \frac{x^m - 1}{x - 1}$. Hence, we have $x^m > P\left(\frac{x^m - 1}{x - 1}\right)$, or $x^m > \frac{Px^m}{x - 1} - \frac{P}{x - 1}$. This inequality will evidently be true, when $x^m = \frac{Px^m}{x - 1}$. Dividing both members of this equation by x^m, and clearing of fractions, we find $x = P + 1$, as enunciated.

It is to be observed that, in this demonstration, we assume $x > 1$.

EXAMPLES.

1. What number substituted for x in the equation, $x^5 - 7x^4 + 6x^3 + 5x^2 + x = 0$, will make the first term greater than the sum of all the other terms? *Ans.* 8.

2. What in the equation, $x^3 - 10x^2 + 12x + 13 = 0$? *Ans.* 14.

3. What quantity in the equation, $x^3 - 2ax^2 + 6ax - 12a = 0$. *Ans.* $1 + 12a$.

Scholium.

In seeking for the positive values of an equation, we need not go beyond the greatest coefficient with a positive sign prefixed to it, plus unity. Thus, in example first, there is no positive value greater than 8, because 8, and all numbers greater than 8, will continually make the first member positive (Art. 463).

Second Limit. — Tenth Property.

466. If we substitute for x in the equation, $x^m + Px^{m-1} + Qx^{m-2} \ldots \ldots Tx + U = 0$, unity, increased by that root of the greatest negative coefficient which is indicated by the number of terms preceding the first negative term, we will have a superior limit of the positive values.

Suppose Nx^{m-n} to be the first negative term, and suppose W to be the greatest negative coefficient. The most unfavourable case obviously is that in which we suppose all the coefficients after N, negative, and equal to W. It is plain, moreover, that any value of x which makes $x^m > W(x^{m-n} + x^{m-n-1} \ldots\ldots + x + 1)$, will, of course, make the first member of the given equation positive. We are to find the value which fulfils this condition.

Resuming the inequality, $x^m > W(x^{m-n} + x^{m-n-1} + x^{m-n-2} \ldots\ldots + x + 1)$, we have, also, $x^m > W\left(\frac{x^{m-n+1} - 1}{x - 1}\right)$, or $x^m > \frac{Wx^{m-n+1}}{x-1} - \frac{W}{x-1}$. And this inequality will be true, when $x^m = \frac{Wx^{m-n+1}}{x-1}$. From which, $x - 1 = \frac{W}{x^{n-1}}$, or $(x-1)x^{n-1} = W$. Now, if $(x-1)(x-1)^{n-1} = W$, then $(x-1)x^{n-1}$ will be greater than W. If, however, we place it equal to W, we will have $(x-1)(x-1)^{n-1} = W$, or $(x-1)^n = W$. Hence, $x = 1 + \sqrt[n]{W}$, which agrees with the enunciation.

It will be seen that in two respects we have taken an unfavorable case. The second limit t en may considerably exceed the greatest positive value, but it is smaller than the first limit, and, therefore, the most used in practice.

EXAMPLES.

1. Required superior positive limit of the values in the equation, $x^5 + 2x^4 - 8x^3 + 7x^2 - 17x + 5 = 0$. *Ans.* $1 + \sqrt{17}$.

2. Required superior positive limit of the values in the equation, $x^4 + 4x^3 + 12x^2 - 5x - 16 = 0$. *Ans.* $1 + \sqrt[3]{16}$.

Eleventh Property.

467. The equation, $x^m + Px^{m-1} + Qx^{m-2} + \ldots\ldots + Tx + U = 0$, can be transformed into another in y, or some other variable, such, that the values of the new unknown quantity shall be some multiple of those of the old unknown quantity.

For, let $y = nx$, then $x = \frac{y}{n}$. Substituting for x, wherever it occurs in the given equation, its value in terms of y, we have $\frac{y^m}{n^m} + P\frac{y^{m-1}}{n^{m-1}} + Q\frac{y^{m-2}}{n^{m-2}} + \ldots T\frac{y}{n} + U = 0$, or $y^m + nPy^{m-1} + n^2Qy^{m-2} \ldots\ldots n^{m-1}Ty + n^mU = 0$. And we see that the transformation is effected by changing x into y, multiplying the coefficient of the second term by the multiple n, that of the third term by n^2, &c. For instance, let it be required to transform the equation, $x^4 + 2x^3 + 4x^2 + 8x + 1 = 0$, into another, in which the values of the new unknown quantity shall be twice as great as those of the old. Then $y = 2x$, and we have $y^4 + 4y^3 + 16y^2 + 64y + 16 = 0$.

Remarks.

The principal use of this transformation is in clearing an equation of fractional coefficients, and at the same time making the new equation preserve the proposed form; that is, the coefficient of the first term is still to be plus unity. Let us take the equation, $x^m + \frac{Px^{m-1}}{m} + \frac{Qx^{m-2}}{n} + \frac{Rx^{m-3}}{mn} \ldots\ldots \frac{Tx}{n} + \frac{U}{n} = 0$, in which we assume mn to be the least common multiple of the denominators. Then $y = mnx$. The transformed equation will then be $\frac{y^m}{(mn)^m} + \frac{Py^{m-1}}{m(mn)^{m-1}} + \frac{Qy^{m-2}}{n(mn)^{m-2}} + \frac{Ry^{m-3}}{mn(mn)^{m-3}} \ldots \frac{Ty}{mn^2} + \frac{U}{n} = 0$, or $y^m + nPy^{m-1} + m^2nQy^{m-2} + m^2n^2Ry^{m-3} \ldots\ldots m^{m-1}n^{m-2}Ty + m^mn^{m-1}U = 0$. And we see that the transformation is effected by changing x into y, and multiplying the second term by the first power of the least common multiple, the third term by the second power of that multiple, &c. Thus, the transformed equation of $x^3 + \frac{x^2}{3} - \frac{x}{4} + 1 = 0$, is $y^3 + 4y^2 - 36y + 1728 = 0$.

468. The two rules just given are, of course, only applicable when the coefficient of the highest power of x is plus unity. When that is not the case, we may either first make this coefficient plus unity, and apply one of the preceding rules, or we may at once make $y = nx$, n in this case representing the product of the least common multiple of the denominators by the coefficient of the highest power of x.

EXAMPLES.

1. Transform the equation, $x^3 - 4x^2 + 2x + 2 = 0$, into another, in which the values shall be twice as great as in the given equation.

Ans. $y^3 - 8y^2 + 8y + 16 = 0$.

2. Transform the equation, $x^4 - 3x^3 + 2x^2 + x + 1 = 0$, into another, whose values are treble those of x.

Ans. $y^4 - 9y^3 + 18y^2 + 27y + 81 = 0$.

3. Transform the equation, $x^4 - \frac{x^3}{2} + \frac{x^2}{3} + \frac{x}{5} + 1 = 0$, into an other, containing only entire coefficients.

Ans. $y^4 - 15y^3 + 300y^2 + 5400y + 810000 = 0$.

4. Transform the equation, $2x^3 - \frac{x^2}{3} + \frac{x}{6} + 1 = 0$, into another, whose coefficients shall all be entire.

Ans. $2y^3 - 2y^2 + 6y + 216 = 0$.

5. Transform the equation, $2x^3 - \frac{x^2}{3} + \frac{x}{6} + 1 = 0$, into another, whose coefficients shall all be entire, and the coefficient of the first term, plus unity. *Ans.* $y^3 - 2y^2 + 12y + 864 = 0$.

In this, make $y = 12x$.

469. When the coefficient of the first term is some whole number different from unity, and the other coefficients are entire, make n, in the equation $y = nx$, equal to the coefficient of the first term. Thus, to transform the equation, $6x^3 - 19x^2 + 19x - 6 = 0$, make $y = 6x$, the transformed equation will be $y^3 - 19y^2 + 114y - 216 = 0$.

Twelfth Property. — Process of Divisors.

470. In every equation in which the coefficient of the first term is unity, and all the other coefficients are entire, a value will divide the last term, the sum arising from adding the quotient so obtained to the

coefficient of the second term from the right, the sum arising from adding this second quotient to the coefficient of the third term from the right, and will thus continue to be an exact divisor of all the successive sums so formed, until the coefficient of the second term from the left has been added to the previous quotient, when the last quotient will be minus unity.

Let us resume the equation,

$$x^m + Px^{m-1} + Qx^{m-2} \ldots\ldots + Tx + U = 0,$$

in which all the coefficients are entire. Suppose a to be a value, the equation will then become, after transposition and division by a, $a^{m-1} + Pa^{m-2} + Qa^{m-3} \ldots\ldots T = -\frac{U}{a}$. Now, since the first member is made up of entire terms, the second member must be entire also. Hence, U is divisible by a, as it ought to be, since U is the product of all the values. Transposing T to the second member, we have $a^{m-1} + Pa^{m-2} + Qa^{m-3} + \&c. = -T - \frac{U}{a} = U'$; U′ representing the algebraic sum of T and $\frac{U}{a}$.

Dividing both members again by a, we get the equation, $a^{m-2} + Pa^{m-3} + Qa^{m-4} + \&c. = -\frac{U'}{a}$.

The first member being entire, the second member must be entire also. Hence, the second condition to be fulfilled by a value is, that it must be an exact divisor of the sum formed by adding the quotient of the last term by the value to the coefficient of the first power of x. This was also to have been anticipated, because the coefficient of the first power of x is made up of the values, taken $m - 1$ and $m - 1$; one of these combinations will not contain the value a, and will have a sign contrary to the first quotient, and will be cancelled by it. The remaining terms making up this coefficient all contain a, hence, a will be a divisor. By transposing the terms in succession, and continuing the division, we will find, after m transpositions and divisions, $\frac{P'}{a} = -1$; P′ representing the sum of the coefficients of x^{m-1} added to the preceding quotient.

Hence, the last condition to be fulfilled by a value is, that it must be an exact divisor of the sum formed by adding the coefficient of x^{m-1} to the preceding quotient.

We, therefore, have the following rule for finding the rational values of an equation.

RULE.

I. *Transform the equation, if necessary, into another, in which all the coefficients shall be entire; that of the first term being plus unity. All the rational values will then be entire* (Art. 459).

II. *Find the superior positive and superior negative limits of the values.*

III. *Write down in succession, in the same horizontal line, all the entire divisors of the last term between zero and these limits.*

IV. *Divide the last term by each of these divisors, and write the respective quotients beneath the corresponding divisors.*

V. *Add the coefficient of the first power of* x *to each of these quotients, and write each sum thus formed beneath the corresponding quotient.*

VI. *Divide these sums by the corresponding divisors in the column of divisors, and write the new quotients under the corresponding sums, rejecting those divisors which give fractional quotients, and crossing out the vertical column in which they occur.*

VII. *Proceed in this way until the coefficient of* x^{m-1} *has been added to the preceding quotient, then those divisors that give minus unity for quotients, are values, and they are the only rational values.*

EXAMPLES.

$x^3 - 2x^2 - 5x + 6 = 0.$

The superior positive limit is $1 + \sqrt[1]{5} = 6.$

The superior negative limit is found by changing $+ x$ into $- x$; the superior positive limit of the transformed equation will then be the superior negative limit of the given equation.

Changing $+ x$ into $- x$, we get $- x^3 - 2x^2 + 5x + 6 = 0$. But the coefficient of the first term must always be plus unity, hence, by multiplying the equation by minus unity we have, $x^3 + 2x^2 - 5x - 6 = 0$.

Then $-(1 + \sqrt{6}) =$ superior negative limit. Assume -4 to be this limit, the square root of 6 being between 2 and 3, we take 3 to be the root, since it is better to have the limit too great than too small. Hence, we have these divisors below the limits, $+1, +2, +3; -1, -2, -3$.

	Positive.	Negative.
Divisors,	$+1, +2, +3,$	$-1, -2,$
Quotients,	$+6, +3, +2,$	$-6, -3,$
Sums,	$+1, -2, -3,$	$-11, -8,$
Quotients,	$+1, -1, -1,$	$+11, +4,$
Sums,	$-1, -3, -3,$	$+9, +2,$
Quotients,	$-1, \times, -1.$	$-9, -1.$

Hence, we have $+1$, $+3$, and -2, for the three values of the given equation.

Let us take, as a second example, the equation, $x^4 - 2x^2 - 8 = 0$. Superior positive limit, $= +9$; superior negative limit, $= -9$. The equation may be written $x^4 \pm 0x^3 - 2x^2 \pm 0x - 8 = 0$.

	Positive.				Negative.			
Divisors,	$+1,$	$2,$	$+4,$	$+8,$	$-1,$	$-2,$	$-4,$	$-8,$
Quotients,	$-8,$	$-4,$	$-2,$	$-1,$	$+8,$	$+4,$	$+2,$	$+1,$
Sums,	$-8,$	$-4,$	$-2,$	$-1,$	$+8,$	$+4,$	$+2,$	$+1,$
Quotients,	$-8,$	$-2,$	$\times,$	$\times,$	$-8,$	$-2,$	$\times,$	$\times,$
Sums,	$-10,$	$-4,$	$\times,$	$\times,$	$-10,$	$-4,$	$\times,$	$\times,$
Quotients,	$-10,$	$-2,$	$\times,$	$\times,$	$+10,$	$+2,$	$\times,$	$\times,$
Sums,	$-10,$	$-2,$	$\times,$	$\times,$	$+10,$	$+2,$	$\times,$	$\times,$
Quotients,	$-10,$	$-1,$	$\times,$	$\times,$	$-10,$	$-1,$	$\times,$	$\times,$

Hence, $+2$ and -2 are values in the given equation. And by dividing the first member of the given equation by the factors $(x - 2)$ and $(x + 2)$ corresponding to these values, we obtain the quotient, $x^2 + 2$. By placing this quotient equal to zero, the remaining two values of the equation will be found to be $+\sqrt{-2}$, and $-\sqrt{-2}$.

471. The process of divisors is applicable to literal, as well as numerical equations, when all the coefficients are entire. A literal equation, too, will best show why it is that the successive sums are divisible by the values.

Take, as an example,

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0.$$

Since the terms are alternately plus and minus, the values must all be positive.

Divisors,	$+a,\ +b,\ +c.$
Quotients,	$-bc,\ -ac,\ -ab.$
Sums,	$ab+ac,\ ab+bc,\ ac+bc.$
Quotients,	$b+c,\ a+c,\ a+b.$
Sums,	$-a,\ -b,\ -c.$
Quotients,	$-1,\ -1,\ -1.$

Hence, the three values are $+a$, $+b$, $+c$.

We will take, as a second example in literal equations,

$$x^3+(a-b+2)x^2+(2(a-b)-ab)x-2ab=0.$$

	Positive.	Negative.
Divisors,	$+a,\ +b,\ +2,$	$-a,\ -b,\ -2,$
Quotients,	$-2b,\ -2a,\ -ab,$	$+2b,\ +2a,\ +ab,$
Sums,	$(2a-ba-4b),\ -b(a+2),\ 2(a-b-ab),$	$a(2-b),\ 4a-b(2+a),\ 2(a-b),$
Quotients,	$\times,\ -(a+2,)(a-b-ab),$	$+(b-2),\ \times,\ -(a-b),$
Sums,	$\times,\ -b,\ 2(a+1)-b(a+1),$	$+a,\ \times,\ +2,$
Quotients,	$\times,\ -1,\ x.$	$-1,\ \times,\ -1.$

Hence, the three values are $+b$, $-a$ and -2.

The process of divisors being of such high practical importance, we will give another example, in which one of the sums is zero.

Take the equation,

$$x^3-x^2-4x+4=0.$$

Superior positive limit, $=5$; superior negative limit, $=-3$.

	Positive.	Negative.
Divisors,	$+1,\ +2,\ +4,$	$-1,\ -2,$
Quotients,	$+4,\ +2,\ +1,$	$-4,\ -2,$
Sums,	$0,\ -2,\ -3,$	$-8,\ -6,$
Quotients,	$0,\ -1,\ \times,$	$+8,\ +3,$
Sums,	$-1,\ -2,\ \times,$	$+7,\ +2,$
Quotients,	$-1,\ -1,\ \times,$	$-7,\ -1.$

Hence, the three values are $+1$, $+2$, and -2.

472. The process of divisors enables us to find all the rational values of any equation. We have only to make all the coefficients entire, if not already so, that of the highest power of x being made plus unity. All the rational values of y in the transformed equation will be entire; find these values, and then those of x will be known from the relation between x and y.

Take $6x^3-19x^2+19x-6=0$. Making $y=6x$, we have the transformed equation in y, $y^3 - 19y^2 + 114y - 216 = 0$. By proceeding as before, we find the three values of y to be $+ 9$, $+ 6$, and $+ 4$. Hence, the corresponding values of x are $\frac{3}{2} + 1$ and $+ \frac{2}{3}$.

473. In this, and in many examples, the last term has a great number of exact divisors, and the process is therefore tedious. It is often convenient to transform an equation, whose last term is too large, into another whose values shall be greater or smaller by a constant quantity. Making $y = z + 2$, in the preceding equation, the transformed equation in z is $z^3 - 13z^2 + 50z - 56 = 0$; and the three values of z are 7, 4, and 2. Hence, those of y are 9, 6, and 4, and those of x, $+ \frac{3}{2}$, $+ 1$, and $+ \frac{2}{3}$, as before.

474. If we solve the equation, $x^4 - 2x^3 - 3x^2 + 8x - 4 = 0$, by the process of divisors, we will get the three values, $+2$, -2, and -1. And, by dividing the first member of the equation by the factors corresponding to these values, we will have left, $x + 1 = 0$. Hence, the value -1 enters twice in the given equation. We see, from this example, that while the process of divisors gives the rational values, it does not show whether any or all of them are repeated once or more. This remark is evidently applicable to all equations whatever. Some test is then necessary, by which we can ascertain the equal values.

EQUAL VALUES.

475. Let the equation be

$$(x - a)^m (x - b)^n (x - c)^p (x - d) (x - e) \ldots\ldots \&c. = 0 \text{ (A)},$$

in which there are m values equal to a, n values equal to b, p values equal to c, and all the other values unequal. Calling D the differential coefficient of this equation, we will have (Art. 366),

$\mathrm{D} = m(x - a)^{m-1} (x - b)^n (x - c)^p (x - d) (x - e)$ &c. $+ n(x - b)^{n-1} (x - a)^m (x - c)^p (x - d) (x - e)$ &c. $+ p(x - c)^{p-1} (x - a)^m (x - b)_n (x - d) (x - c)$ &c. $+ (x - a)^m (x - b)^n (x - c)^p (x - e)$ &c. $+ (x - a)^m (x - b)^n (x - c)^p (x - d)$ &c., plus other terms (B).

It is plain that the greatest common divisor, D′, between (A) and (B), will be

$$\mathrm{D}' = (x - a)^{m-1} (x - b)^{n-1} (x - c)^{p-1}.$$

We see that this divisor contains $(m - 1)$ values equal to a, $(n - 1)$ values equal to b, and $(p - 1)$ values equal to c. Hence, it is plain that the number of equal values in each set of the greatest common divisor is one less than in the giv equation. Therefore, to ascertain whether there are equal values, we have this

RULE.

Find the differential coefficient of the first member of the given equation (or, as it is generally called, the first derived polynomial), next get the greatest common divisor between the first member of the proposed equation and this differential coefficient, place this common divisor equal to zero, and find its values. Each independent value so found will be repeated once oftener in the given equation than in the greatest common divisor; if the latter, for instance, contain two values equal to a, *and four values equal to* b, *the former will contain three values equal to* a, *and five equal to* b. *If the greatest common divisor be of too high a degree to solve, we may find the second differential coefficient, or second derived polynomial of the given equation, then the greatest common divisor between this polynomial and the first member of the given equation. Each independent value of this common divisor will be repeated twice oftener in the given equation.*

EXAMPLES.

1. Does the equation, $x^4 - 2x^3 + \frac{3}{2}x^2 - x + \frac{1}{2} = 0$, contain equal values?

First derived polynomial, $4x^3 - 6x^2 + 3x - 1$.

Greatest common divisor, $x - 1$, which, placed equal to zero, gives $x = 1$. Hence, the given equation contains two values equal to 1. If we divide that equation by $(x-1)^2$, we will have left, $x^2 + \frac{1}{2} = 0$. Hence, the other two values are $+\sqrt{-\frac{1}{2}}$, and $-\sqrt{-\frac{1}{2}}$.

2. Does the equation, $x^5 - x^4 - 14x^3 - 26x^2 - 19x - 5 = 0$, contain equal values?

First derived polynomial, $5x^4 - 4x^3 - 42x^2 - 52x - 19$.

Greatest common divisor, $(x+1)^3$. Hence, the given equation contains four values equal to -1, and, by dividing the equation by $(x+1)^4$, we find the other value to be $+5$.

3. Required the equal values in the equation, $x^4 - 2x^2 + 1 = 0$.

Ans. Two values $= +1$ and two values $= -1$.

4. Required all the values of the equation, $x^5 - \frac{x^4}{2} - 2x^3 + x^2 + x - \frac{1}{2} = 0$. *Ans.* $+1, +1, +\frac{1}{2}, -1$ and -1.

5. Required all the values of the equation, $x^4 + x^3 - 6x^2 - 4x + 8 = 0$. *Ans.* $+1, +2, -2, -2$.

After dividing out the equal factors, $(x+2)(x+2)$, we had a quadratic left, which was solved by known rules.

6. Find the values of the equation, $x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = 0$. *Ans.* $+1, +1, +1, -1$, and -1.

The first greatest common divisor is $x^3 - x^2 - x + 1$. As this, when placed equal to zero, is too high to solve, we find the greatest common divisor between the second differential coefficient of the given equation, and the first member of that equation, or, what amounts to the same thing, the greatest common divisor between $x^3 - x^2 - x + 1$, and its derived polynomial, $3x^2 - 2x - 1$. This common divisor is $(x-1)^2$. Hence, the given equation contains *three* values equal to plus one, and, by dividing out by the factors, $(x-1)(x-1)(x-1)$, or, $x^3 - 3x^2 + 3x - 1$, we have left, $x^2 + 2x + 1 = 0$. Hence, the other two values are, -1 and -1.

7. Solve the equation, $x^5 + x^4 - 5x^3 - x^2 + 8x - 4 = 0$.
Ans. $x = +1, +1, +1, -2, -2$.

8. Find the five values of the equation, $x^5 + 2x^4 - 16x - 32 = 0$.
Ans. $+2, -2, -2, +\sqrt{-4}, -\sqrt{-4}$.

9. Find the values of the equation, $x^6 + x^5 - 2x^4 - 16x^2 - 16x + 32 = 0$. *Ans.* $+1, +2, -2, -2, +\sqrt{-4}, -\sqrt{-4}$.

10. Find the values of the equation, $x^3 - (2a+b)x^2 + (a^2 + 2ab)x - a^2b = 0$. *Ans.* $+a, +a, +b$.

11. Solve the equation, $x^4 - 2(a+b)x^3 + (a^2 + 4ab + b^2)x^2 - 2(a^2b + ab^2)x + a^2b^2 = 0$. *Ans.* $+a, +a, +b, +b$

DERIVED POLYNOMIALS.

476. We have had occasion to use the relation, $x = a + y$, in transforming the general equation in x into another in y, such, that the second term was wanting. But the transformation in terms of y and a is of more general application. Sometimes a is an undetermined constant, whose value is afterwards determined in such a manner as to make the equation fulfil a given condition. Sometimes a is a determinate number or quantity, which expresses the difference between the values of the given equation and another which we wish to form. Suppose, for instance, we wished to transform the equation, $x^3 - x^2 + 4x + 5 = 0$, into another in y, such, that each value of x should exceed by 2 each corresponding value of y. Then, a must be made equal to 2 in the relation, $x = a + y$, and we must substitute $2 + y$ for x wherever it occurs. Hence, the equation in y would be $(2 + y)^3 - (2 + y)^2 + 4(2 + y) + 5 = 0$, or, expanding and reducing, $y^3 + 5y^2 + 12y + 17 = 0$. In this case, the transformation is easily effected by actually substituting the value of x in terms of y, and developing the several binomials in the new equation. But, when the given equation is of a high degree, this process is tedious and impracticable. Suppose, for example, it were required to transform the equation, $x^{12} - 40x^{11} + x^{10} - 2x^9 + 8x^8 + 5x^7 - 4x^6 + 9x^6 + 12x^4 - x^2 + 3x + 15 = 0$, into another in y, such, that the values of x should be less by unity than the corresponding values of y. Then, $a = -1$ in the relation, $x = a + y$, and we must substitute for x, wherever it occurs, $(y - 1)$. It is plain that the development of the several binomials in the new equation will be exceedingly tedious.

The method of derived polynomials enables us to effect the required transformation in y *without substitution and development.*

Let us resume the equation,

$$x^m + Px^{m-1} + Qx^{m-2} \ldots\ldots + Tx + U = 0,$$

and let us suppose, as before, $x = a + y$.

The new equation in y will then be

$$(a + y)^m + P(a + y)^{m-1} + Q(a + y)^{m-2} \ldots\ldots T(a + y) + U = 0.$$

Let us assume

$$(a + y)^m + P(a + y)^{m-1} + Q(a + y)^{m-2} \ldots\ldots T(a + y) + U = A + By + Cy^2 + Dy^3 + Ey^4 + \&c.$$

Making $y=0$, we get

$$a^m + Pa^{m-1} + Qa^{m-2} \ldots\ldots + Ta + U = A.$$

Differentiating the given equation, and dividing by dy, we get

$$m(a+y)^{m-1} + P(m-1)(a+y)^{m-2} + Q(m-2)(a+y)^{m-3} \ldots\ldots + T = B + 2Cy + 3Dy^2 + 4Ey^3 \text{ (R).}$$

Now, make $y=0$, and denote by A′ what B becomes in that case, we get

$$ma^{m-1} + (m-1)Pa^{m-2} + (m-2)Qa^{m-3} \ldots\ldots + T = B = A'.$$

Differentiating (R), we get

$$m(m-1)(a+y)^{m-2} + (m-1)(m-2)P(a+y)^{m-3} + (m-2)(m-3)Q(a+y)^{m-3} + \&c. = 2C + 2 \cdot 3Dy + 3 \cdot 4Ey^2 + \&c. \text{ (S).}$$

Making $y=0$, and representing by A″ what 2C becomes, we get

$$m(m-1)a^{m-2} + (m-1)(m-2)Pa^{m-3} + (m-2)(m-3)Qa^{m-3} + \&c. = A'' = 2C.$$

$$\text{Then, } C = \frac{A''}{2} = \frac{m(m-1)a^{m-2} + (m-1)(m-2)Pa^{m-3} + \&c.}{2}$$

Differentiating (S), we get

$$m(m-1)(m-2)(a+y)^{m-3} + (m-1)(m-2)(m-3)P(a+y)^{m-4} + (m-2)(m-3)(m-4)Q(a+y)^{m-5} + \&c. = 2 \cdot 3 \cdot D + 2 \cdot 3 \cdot 4Ey + \&c.$$

Making $y=0$, and preserving a similar notation, we have

$$D = \frac{A''}{2 \cdot 3} = \frac{m(m-1)(m-2)a^{m-3} + (m-1)(m-2)(m-3)Pa^{m-3}}{2 \cdot 3} + \frac{(m-2)(m-3)(m-4)Qa^{m-5}}{2 \cdot 3} + \&c.$$

By proceeding in the same way, all the other coefficients could be determined. But the law of formation is already apparent, A is what the given equation becomes when a (or the difference between x and y) is substituted for x; A′ is formed from A, by multiplying the coefficient of a in every term by its exponent, and then diminishing that exponent by unity; $\frac{A''}{2}$ is formed in precisely the same way, except that the result is divided by 2; $\frac{A'''}{3}$ is formed in like manner from A″, except that the result is divided by 3.

$\frac{A''''}{4}$ is formed according to the same law from A‴, the result being divided by 4.

Hence, equation (P) becomes

$$(a+y)^m + P(a+y)^{m-1} + Q(a+y)^{m-2} \ldots\ldots + T(a+y) + U =$$

$$a^m + Pa^{m-1} + Qa^{m-2} \ldots\ldots + Ta + U +$$

$$(ma^{m-1} + (m-1)\,Pa^{m+2}(m-2)\,Qa^{m-3} \ldots\ldots + T)\,y +$$

$$\left(\frac{+\,m(m-1)\,a^{m-2} + (m-1)\,(m-2)\,Pa^{m-3} + (m-2)\,(m-3)\,Qa^{m-4} + \&c.}{1\,.\,2}\right)y^2 +$$

$$\left(\frac{m(m-1)\,(m-2)\,a^{m-3} + (m-1)\,(m-2)\,(m-3)\,Pa^{m-4} + (m-2)\,(m-3)\,(m-4)\,Qa^{m-5} + \&c.}{1\,.\,2\,.\,3}\right)y^3 +$$

$$\left(\frac{m(m-1)(m-2)(m-3)a^{m-4} + (m-1)(m-2)(m-3)(m-4)Pa^{m-5}(m-2)(m-3)(m-4)(m-5)Qa^{m-6}}{1\,.\,2\,.\,3\,.\,4}\right)y^4 + \&c. + \&c.$$

The second member is also equal to $A + A'y + \frac{A''y^2}{1\,.\,2} + \frac{A'''y^3}{1\,.\,2\,.\,3} + \&c.$ Hence, since the second member of the equation in x was zero, the transformed equation in y may be written, $A + A'y + \frac{A''y^2}{1\,.\,2} + \frac{A'''y^3}{1\,.\,2\,.\,3} + \frac{A^{iv}y^4}{1\,.\,2\,.\,3\,.\,4} + \&c. = 0$. The expressions, A', A'', A''', are called derived polynomials.

A' is the first derived polynomial.

A'', the second derived polynomial.

A''', the third derived polynomial, &c. &c.

477. To transform an equation into another, in which the values of the new unknown quantity shall differ from those of the old by some constant quantity, we have then this

RULE.

I. *Replace* x *in the given equation, wherever it occurs, by the assumed difference, and call the result* A.

II. *Find the derived polynomial of* A, *and call it* A'.

III. *Find the derived polynomial of* A', *and call it* A''.

IV. *Find all the other derived polynomials in the same, and calculate their values.*

V. *Substitute the values so found in the formula,* $A + A'y + \frac{A''y^2}{1 \cdot 2} + \frac{A'''y^3}{1 \cdot 2 \cdot 3} + \&c. = 0$, *and the transformed equation will be found.*

EXAMPLES.

1. Transform the equation, $x^3 - x^2 + 4x + 5 = 0$, into another in y, so that the values of x shall exceed those of y by 2.

Then we have, by the rule,

$$\begin{aligned}
(2)^3 - (2)^2 + 4\,(2) + 5 &= 17 = A,\\
3\,(2)^2 - 2\,(2) + 4 &= 12 = A,'\\
6\,(2) - 2 &= 10 = A'',\\
6 &= \ \ 6 = A''',\\
0 &= \ \ 0 = A^{iv}.
\end{aligned}$$

Hence, the formula, $A + A'y + \frac{A''y^2}{1 \cdot 2} + \frac{A'''y^3}{1 \cdot 2 \cdot 3} + \&c. = 0$, gives $17 + 12y + 5y^2 + y^3 = 0$; or, changing the order of the terms, $y^3 + 5y^2 + 12y + 17 = 0$, the same as before found.

2. Transform the equation, $x^6 - 5x^5 + 4x^2 + x + 1 = 0$, into another in y, so that the values of x shall exceed those of y by unity.

$$\begin{aligned}
&(1)^6 - 5\,(1)^5 + 4\,(1)^2 + (1)^1 + 1 = 2 = A,\\
&6\,(1)^5 - 25\,(1)^4 + 8\,(1) + 1 = -10 = A',\\
&30\,(1)^4 - 100\,(1)^3 + 8 = -62 = A'',\\
&120\,(1)^3 - 300\,(1)^2 = -180 = A''',\\
&360\,(1)^2 - 600\,(1) = -240 = A^{iv},\\
&720\,(1) = 720 = A^{v},\\
&720 = 720 = A^{vi}.
\end{aligned}$$

Hence, the transformed equation is

$$2 - 10y - 31y^2 - 30y^3 - 10y^4 + 6y^5 + y^6 = 0,$$
$$\text{or, } y^6 + 6y^5 - 10y^4 - 30y^3 - 31y^2 - 10y + 2 = 0.$$

3. Transform the equation, $x^4 - 8x^3 + 9x^2 + x + 12 = 0$, into another, whose second term shall be wanting.

Ans. $y^4 - 15y^2 - 27y + 2 = 0.$

In this example, $x = y - (\frac{-8}{4}) = y + 2$ (Art. 455).

4. Transform the equation, $2x^3 + x^2 + x - 4 = 0$, into another, whose second term shall be wanting. *Ans.* $y^3 + \frac{5}{12}y - \frac{448}{216} = 0.$

First divide the given equation by 2, to bring it under the proposed form. Then, make $x = y - \frac{\frac{1}{2}}{3} = y - \frac{1}{6}$.

5. Transform the equation, $-x^3 + x^2 + x - 1 = 0$, into another, whose second term shall be wanting. *Ans.* $y^3 - \frac{4}{3}y + \frac{16}{27} = 0.$

First bring the equation under the proposed form by multiplying by minus unity. Then, make $x = y + \frac{1}{3}$.

6. Transform the equation, $x^6 - 18x^5 + 135x^4 + x^3 - x^2 + 2x + 4 = 0$, into another, whose second term shall be wanting.

Ans. $y^6 + 541y^3 + 3653y^2 + 8771y + 7318 = 0.$

Why did the second and third terms disappear together?

7. Transform the equation, $x^7 - 7x^6 + x^5 - 2x^4 + x^3 - 3x^2 + x + 8 = 0$, into another in y, such, that the values of y shall be less by unity than those of x.

Ans. $y^7 - 20y^5 - 67y^4 - 102y^3 - 86y^2 - 40y = 0.$

Why is the absolute term wanting in the transformed equation? Why did the second term disappear?

8. Transform the equation, $4x^3 - 3x^2 + 2x - 3 = 0$, into another in y, such, that the values of y shall be greater by 2 than those of x.

Ans. $4y^3 - 27y^2 + 62y - 51 = 0.$

It will be seen that $+1$ is a value of x in the given equation, and $+3$ is a value of y in the transformed equation, as it ought to be.

9. Transform the equation, $4x^4 - 3x^3 + 2x^2 - 3x = 0$, into another in y, such, that the values of y shall be less by unity than those of x.

Ans. $4y^4 + 13y^3 + 17y^2 + 10y = 0.$

Why is the absolute term wanting in the transformed equation? Why are all the terms positive?

10. Transform the equation, $5x^5 - 50x^4 + 200x^3 + 9x^2 + x - 165 = 0$, into another in y, such, that the values of y shall be less by 2 than those of x. *Ans.* $5y^5 + 409y^2 + 1237y + 838 = 0$.

Why did the second and third terms disappear?

EQUATION OF DIFFERENCES.

478. We have now shown how all the rational values could be obtained, either by operating directly upon the given equation, when of the proposed form, or upon the transformed equation, and then, by means of the relations between the values in the transformed and given equations, ascertain the rational values in the given equation. If we would next divide the given equation by the factors corresponding to its rational values, and place the quotient resulting equal to zero, we would have a new equation containing only irrational and imaginary values. Before explaining the method of finding the irrational values, it becomes necessary to show how to find an equation, whose values shall be equal to the difference of values in the given equation. Such an equation is called *the equation of differences.*

Let x', x'', x''', &c., be values in the given equation, and y a value in the equation of differences sought. It is plain that this equation will be of the same form, whatever pair of values we assume in the given equation. Hence, we may assume $y = x'' - x'$, then $x'' = y + x'$. Since x'' is, by hypothesis, a value in the given equation, $x^m + Px^{m-1} + Qx^{m-2} \ldots\ldots + Tx + U = 0$, by substitution, we will have $x''^m + Px''^{m-1} + Qx''^{m-2} \ldots\ldots Tx'' + U = 0$. Now, if we replace x'' by $y + x'$ in this equation, and develop by the formula in Article 477, the new equation will become $X_0 + X_1 y + X_2\frac{y^2}{1 \cdot 2} + X_3 \frac{y^3}{1 \cdot 2 \cdot 3} + \&c. = 0$; in which X_0 denotes what the given equation becomes when x' is substituted for x, X_1 is the derived polynomial of X_0; X_2 the derived polynomial of X_1, &c., &c. But, since x' is, by hypothesis, a value, X_0 is equal to zero, and, by dividing by y, the new equation becomes $X_1 + X_2\frac{y}{1 \cdot 2} + X_3\frac{y^3}{1 \cdot 2 \cdot 3} + \&c. = 0$ (D), and is the equation, which combined with the given equation, will give the one sought.

479. It is obvious that, to obtain (D), we have only to get the derivative of the first member of the given equation, add to this the second derived polynomial, multiplied by the first power of y, divided by 1 . 2, plus the third derived polynomial, multiplied by y^2, divided by 1 . 2 . 3, plus other derived polynomials, multiplied by corresponding powers of y with their appropriate divisors.

As a simple illustration, let it be required to find the equation of differences to the equation, $x^2 - 4 = 0$. Then $X_1 = 2x$, $X_2 = 2$, and (D) becomes $2x + y = 0$. Eliminating x between these equations, by the method of the greatest common divisor, we have,

$$\begin{array}{l|l}
4x^2 - 16 & 2x + y \\
\underline{4x^2 + 2xy} & 2x - y = \text{quotient.} \\
\quad - 2xy - 16 & \\
\quad \underline{- 2xy - y^2} & \\
\qquad y^2 - 16 = 0, \text{ equation of differences.} &
\end{array}$$

The values of y in this equation, are $+ 4$ and $- 4$, and those of x in the given equation, $+ 2$ and $- 2$, the difference between which is either $+ 4$ or $- 4$.

If in the equation, $y^2 - 16 = 0$, we make $y^2 = z$, we will have $z - 16 = 0$, and the value of z will be equal to the square of the difference of the values of x. Such an equation is called the *equation of the square of the differences.*

The degree of the equation of differences will be expressed by the number of combinations of m values, taken two and two, or by $m(m - 1)$, and, since this product is always even whatever may be the value of m, the sought equation will always be of an even degree.

As a second illustration, let it be required to find the equation of differences to the equation, $x^3 - x - 6 = 0$. Then, $X_1 = 3x^2 - 1$, $X_2 = 6x$, $X_3 = 6$, X_4, X_5, &c. $= 0$, and equation (D) becomes $3x^2 - 1 + 3xy + y^2 = 0$; or, $3x^2 + 3xy + y^2 - 1 = 0$; and, preparing the given equation for division by multiplying by y, we have

$$\begin{array}{l|l} 3x^3 - 3x - 18 & 3x^2+3x\,y+y^2-1 \\ \underline{3x^3+3x^2y+xy^2-x} & x-y = \text{Quotient.} \end{array}$$

$$-3x^2y-(y^2+2)x-18$$
$$\underline{-3x^2y-3xy^2-y^3+y}$$
$$2(y^2-1)\,x + y^3 - y - 18$$

$$3x^2+3xy+y^2-1$$
$$\underline{\qquad 2(y^2-1)}$$
$$\begin{array}{l|l} 6(y^2-1)x^2+6(y^2-1)xy+2(y^2-1)^2 & \text{2d Quotient} = 3x. \end{array}$$
$$\underline{6(y^2-1)x^2+3y^3x-3xy-54x}$$
$$3(y^2-1)xy+54x+2(y^2-1) = \text{2d Remainder.}$$

$$\text{or, } 3(y^3-y+18)x+2(y^2-1)^2 \qquad \text{Multiplying by } 2(y^2-1)$$
$$\underline{\qquad 2(y^2-1)}$$

$$\begin{array}{l|l} 6(y^3 - y + 18)\,x(y^2 - 1) + 4(y^2 - 1)^3 & 2(y^2-1)x+y^3-y-18 \\ \underline{6(y^3-y+18)x(y^2-1)+3(y^3-y+18)(y^3-y-18)} & 3(y^3-y+18) = \text{3d Quotient.} \\ 4(y^2-1)^3-3\,(y^3-y+18)\,(y^3-y-18)=0 & \end{array}$$

or, developing $4(y^6 - 3y^4 + 3y^2 - 1) - 3y^6 + 3y^4 + 54y^3 + 3y^4 - 3y^2 - 54y - 54y^3 + 54y + 3(18)^2 = 0$; or, reducing $y^6 - 6y^4 + 9y^2 + 968 = 0$, which is the equation of differences sought.

Now, make $y^2 = z$, and we have for the equation of the square of the differences $z^3 - 6z^2 + 9z + 968 = 0$.

In the above development, the alternate terms were struck out. This was to have been anticipated from what had been said. But we may show, in another manner, that the equation of differences can contain only even powers of y.

For, let a, b, c, d, &c., be the numerical differences of the values of the given equation, then, $+a$, $-a$, $+b$, $-b$, &c., will be values in the equations of differences; and, since the factors corresponding to these values, permuted in pairs, give us $(y+a)\,(y-a)$, $(y+b)\,(y-b)$, &c., the first member of the equation of differences will be, $(y^2 - a^2)\,(y^2-b^2)\,(y^2 - c^2)$, &c.

Now, making $y^2 = z$, we will have $(z - a^2)\,(z - b^2)\,(z - c^2)$ &c. $= 0$ for the equation of the square of the differences. The degree of the last equation will only be half as great as that of the equation of differences.

EXAMPLES.

1. Given, $x^2 - 9 = 0$, to find the equation of differences.

Ans. $y^2 - 36 = 0$.

2. Given, $x^3 + 9x + 4 = 0$, to find the equation of differences.

Ans. $y^6 + 54y^4 + 81y^2 + 3348 = 0$.

After the first division the remainder will be, $2(y^2 + 9)\,x + y^3 + 9y + 12$. Multiply the last divisor twice by $2(y^2 + 9)$. Then, after two divisions, you will have $4(y^2 + 9)^3 - 3(y^3 + 9y - 12)\,(y^3 + 9y + 12)$, which will reduce to the expression above.

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3. Given, $x^3 + ax + b = 0$, to find the equation of differences.

Ans. $y^6 + 6ay^4 + 9a^2y^2 + 4a^3 + 27b^2 = 0$.

First remainder is, $2(y^2 + a^2)x + y^3 + ay + 3b$. Use $2(y^2 + a^2)$ twice as a multiplier.

4. Given, $x^3 - x = 0$, to find the equation of differences.

Ans. $y^6 - 6y^4 + 9y^2 - 4 = 0$.

First remainder is, $2(y^2 - 1)x + y^3 - y$. The coefficient of x is used twice as a multiplier.

It will be seen that $+1$ and -1 are values in the equation of differences. Dividing by $y^2 - 1$, and solving the resulting equation, $y^4 - 5y^2 + 4 = 0$, by the rules for binomial equations, we get the four values, $+2$, -2, $+1$ and -1. The six values, then, in the equation of differences, are, $+2$, -2, and $+1$ and -1, repeated twice. These ought to be the values, since those in the given equation are, 0 $+1$ and -1, the differences between which are those given above.

IRRATIONAL VALUES.

480. An equation, freed from the factors corresponding to the rational values, will contain only irrational or imaginary values, or both irrational and imaginary values. We can best explain the process of finding the irrational values by an example.

Let us take the equation, $x^5 - 2x^3 - 2 = 0$, to find one positive rational value. The superior positive limit is, $1 + \sqrt{2} = 3$. Substituting the natural numbers from 0 up to 3, we find that 0 and 1 give negative results when substituted for x in the equation, and that 2 gives a positive result. Hence, a value of the equation lies between 1 and 2, and 1 is the entire part of the irrational value.

Now, make $x = y + 1$ in the given equation, then the new equation in y will have values less by unity than those in x. Hence, y will contain the decimal part of the value of x. This transformation can be most readily effected by the formula of Article 477. We have

$$(1)^5 - 2(1)^3 - 2 = -3 = A.$$
$$5(1)^4 - 6(1)^2 = -1 = A'.$$
$$20(1)^3 - 12(1)^1 = 8 = A''.$$
$$60(1)^2 - 12 = 48 = A'''.$$
$$120(1)^1 = 120 = A^{iv}.$$
$$120 = 120 = A^{v}.$$

Then, $A + A'y + \frac{A''y^2}{1 \cdot 2} + \frac{A'''y^3}{1 \cdot 2 \cdot 3} + \frac{A^{iv}y^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{A^{v}y^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ $+$ &c. $= 0$ becomes, $-3 - y + 4y^2 + 8y^3 + 5y^4 + y^5 = 0$; or, changing the order of terms, $y^5 + 5y^4 + 8y^3 + 4y^2 - y - 3 = 0$.

If, now, we make $z = 10y$, then $y = \frac{z}{10}$. The first figure, then, of the value of z, in the transformed equation, will be *tenths* in the value of y; and, consequently, tenths in the value of x. The transformed equation (Article 467) is,

$$z^5 + 50z^4 + 800z^3 + 4000z^2 - 10000y - 300000 = 0.$$

$$\text{S. P. Limit} = 1 + \sqrt[4]{300000} = 1 + 23 = 24.$$

The limit is here too great for any practical use. Substituting, between 0 and 10, we find a change of sign in the results corresponding to the substitution of 5 and 6. For 5, we have,

$$(5)^5 + 50(5)^4 + 800(5)^3 + 4000(5)^2 - 10000(5) - 300000 = -115625.$$

And 6 gives,

$$(6)^5 + 50(6)^4 + 800(6)^3 + 4000(6)^2 - 10000(6) - 300000 = +29376.$$

Hence, 5 is the entire part of the value of z, and this corresponds to five-tenths in y and x. Therefore, 1 . 5 is the approximate value of x, to within tenths. To get a nearer approximation, let us transform the equation in z into another in w, so that the values of w shall be less by 5 than those of z. Then w will contain the remaining part of the decimal value of z. The equation in w is,

$$w^5 + 75w^4 + 2050w^3 + 24750w^2 + 118125w - 115625 = 0.$$

Making $t = 10w$, or $w = \frac{t}{1}$, the transformed equation in t (Art. 467), will be,

$$t^5 + 750t^4 + 205000t^3 + 24750000t^2 + 1181250000t - 11562500000 = 0.$$

On trial, we find that 8 and 9 give results with contrary signs; hence, 8 is the entire part of the value of t, and corresponds to ·8 in w, and to .08 in y and x. We have, then, for the approximate value of x, 1·58. The process would be, in all respects, the same, were a negative value to be found, except that we would find the negative limits.

Let us take, as a second example, $x^5 - 10x^3 + 6x + 1 = 0$.

We find that 0 and 1 give results with contrary signs, and so do 3 and 4. Let us find the decimal part of the value, whose entire part is 3. Make $x = y + 3$. The equation in y will be,

$$y^5 + 15y^4 + 80y^3 + 180y^2 + 141y - 8 = 0,$$

and that in z,

$$z^5 + 150z^4 + 8000z^3 + 180000z^2 + 1410000z - 800000 = 0.$$

Substituting 0 and 1, we find a change of sign. Hence, 0 is the tenths of the given equation. Making $z = 0 + w$, we have,

$$w^5 + 150w^4 + 8000w^3 + 180000w^2 + 1410000w - 800000 = 0.$$

The equation in t, is,

$$t^5+1500t^4+800000t^3+180000000t^2+14100000000t-80000000000=0.$$

We find that 5 and 6, when substituted, give a change of sign.

Now, make $t = s + 5$, the equation in s will be,

$$s^5+175s^4+830250s^3+192226280s^2+15960753125s-4899059375=0.$$

By changing this into an equation in r, making $r = 10s$, we will find a change between 3 and 4. Hence, 3·053 is the approximate value of x to within *thousandths.*

481. In this process we have proceeded upon the hypothesis, that but one real root lay between two successive integers. To ascertain whether this is the case, we have only to transform the given equation in x into another in y, so that the values of y shall be the squares of the differences of those of x. Next, find the inferior limit of the positive values of y. Suppose D^2 to be this limit; then, since D^2 is less than the least value in the *equation of the* square of the differences, $\sqrt{D^2}$, or D, will be less than the least difference between the values in the given equation. Now, if D be > 1, it is plain that no real root will be comprised between two successive integers, and the process described above can be pursued. A similar course of reasoning can be applied to D'^2, the inferior limit of the negative values in the equation of the squares of the differences.

But, if $D < 1$, then two or more real roots may be comprised between two consecutive integers. In this case, we have only to substitute a series of numbers, whose common difference shall be $=$ or $<$ D. Then, those numbers, which give results with contrary signs, will have but one real value between them. Another method of frequent appli·

cation, when D is a proper fraction, $\frac{1}{r}$, is to transform the equation in x into another in y, by making $x = \frac{y}{r}$. Then, the differences of the values of y will be greater than unity, and only one real root will lie between the successive integers in the transformed equation. For, let x' and x'' be consecutive values of x, then $x' = \frac{y'}{r}$, and $x'' = \frac{y''}{r}$. Hence, $(x' - x'')\, r = y' - y''$. Then the differences between the consecutive values of y is r times greater than between those of x, and, as r is the denominator of D, $y' - y''$ must be greater than unity.

EXAMPLES.

1. Find one irrational value in the equation, $x^5 - 8x^3 + 7x^2 - 56 = 0$.
Ans. 2·828.

2. Find one irrational value in the equation, $x^4 + 3x^3 - 4x^2 - 15x - 5 = 0$.
Ans. 2·236.

3. Find one irrational value in the equation, $x^5 + 2x^3 - 2x^2 - 4 = 0$.
Ans. 1·264.

4. Find one irrational value in the equation, $x^3 - 7x + 7 = 0$.
Ans. —3·048.

5. Find one irrational value in the equation, $x^4 + x^3 - 12x^2 - 17x - 85 = 0$.
Ans. 4·123.

6. Find one irrational value in the equation, $x^2 - 2 = 0$.
Ans. 1·414.

7. Find, by the process of irrational values, one value in the equation, $x^2 - 4 = 0$.
Ans. $x = 2$, the decimal part in all the transformed equations being zero.

8. Find one irrational value in the equation, $x^4 + x^3 - 25x^2 - 26x - 26 = 0$.
Ans. 5·099.

482. If we apply the foregoing process to an example of the form, $x^4 + 14x^2 - 49 = 0$, the consecutive numbers will give no change of sign. One value in the equation of differences will be found to be zero, and also one in the equation of the square of the differences.

D^2 and D are then both zero. When this is the case, we may infer the presence of equal values. On trial, as in Art. 475, we find $x^2 - 7$ the greatest common divisor between the first member of the given equation and its derived polynomial. We see, by this example, that the preceding method is only applicable to equations which have been freed from their equal values.

While the foregoing method affords a complete solution to the problem of finding the irrational values of numerical equations, yet it is of difficult application to equations of high degrees, and, in all equations, whether of low or high degrees, the difficulty increases with the number of decimal places sought.

NEWTON'S METHOD OF APPROXIMATION.

483. This is known as the *method of successive substitutions*, and consists in substituting, in the given equation, the natural numbers between the limits, until results with contrary signs are obtained. Let a be the least of two consecutive numbers which give results with contrary signs. Then a is the first figure, or entire part of the value sought. Substitute $a + y$ for x in the given equation, y being a small fraction, whose second and higher powers may be neglected. Hence, y may be found in the transformed equation, and $a + y$ will constitute the first approximation to the value of x. Let $a + y$ be denoted by b, and make $x = b + y'$, y' being a small fraction, whose higher powers may be neglected. The transformed equation will give the value of y', and this, with b, will constitute the second approximation to x. Calling $b + y'$, c, and making $x = c + y''$, we can get a third approximation to the value of x, and may thus carry the approximation to as many places of decimals as we please.

Let us take, as an illustration, the equation, $x^2 - 2 = 0$. Substituting between the limits, we find that 1 and 2 give different signs. Then 1 is the entire part of the value. Make $x = 1 + y$, and reject y^2, we will have $1 + 2y - 2 = 0$, or $y = \frac{1}{2} = \cdot 5$. Hence, for first approximation, $x = 1\cdot 5$. Now, make $x = 1\cdot 5 + y'$, and we will have, after rejecting y'^2, $2\cdot 25 + 3y' - 2 = 0$, or, $y' = -\dfrac{\cdot 25}{3} = -\cdot 0833$.

Then, for the second approximation, we get $x = 1\cdot 5 - \cdot 0833 = 1\cdot 4167$. Place $x = 1\cdot 4167 + y''$, we get

$$2\cdot 8834y'' = -\cdot 00303889, \text{ or, } y'' = -\cdot 00107.$$

Then, for a third approximation, $x = 1\cdot 4167 - \cdot 00107 = 1\cdot 41563$.

484. The accuracy of the process evidently depends upon the unknown quantity introduced being a small fraction. Whenever, then, the substitution of the natural numbers make y an improper fraction, more minute substitutions must be made, unless we intend to carry the approximation to several places of decimals.

Thus, take the example, $x^3 + x - 8 = 0$, we find that 1 and 2 give results with contrary signs. Then, 1 is the entire part of the value sought. Making $x = y + 1$, we have, $y^3 + 3y + 3y^2 + 1 + y + 1 - 8 = 0$, or, rejecting the higher powers of y, $4y - 6 = 0$. Hence, $y = \frac{6}{4} = 1{\cdot}5$, an improper fraction. Then, for the first approximation, we have, $x = 2{\cdot}5$, a result obviously too great.

Next, place $x = 2{\cdot}50 + y'$, we have, after rejecting higher powers of y', $(2{\cdot}50)^3 + 3\,(2{\cdot}50)^2y + 2{\cdot}50 + y - 8 = 0$. Then, $y' = -{\cdot}512$, and, for the second approximation, $x = 1{\cdot}988$. Making now $x = 1{\cdot}988 + y''$, we will have, after rejecting the higher powers of y'',

$$(1{\cdot}988)^3 + 3\,(1{\cdot}988)^2y'' + 1{\cdot}988 + y'' - 8 = 0,$$
$$\text{or, } 7{\cdot}856862272 + 11{\cdot}856432y'' + 1{\cdot}988 + y'' - 8 = 0,$$
$$\text{or, } 11{\cdot}856432y'' = -1{\cdot}844862272.$$

Hence, $y'' = -{\cdot}144$ nearly, and, for a third approximation, $x = 1{\cdot}844$, which differs from the true value by less than one hundredth. We see, from this example, that the error was considerable when y was an improper fraction, but became reduced by carrying the approximation farther. A closer approximation could have been obtained, without carrying the operation so far, by making minute substitutions. A change of sign would have been found between the results corresponding to 1·8 and 1·9.

LAGRANGE'S METHOD.

485. This differs from Newton's, in that the unknown expression added to complete each successive value is fractional in form, and in that the transformation is made into an equation involving this unknown quantity. Thus, let a be the entire part or number next below the value, we make $x = a + \frac{1}{y}$, and get a transformed equation in y, such, that the values of y must be greater than unity. Let b be the entire part of the value of y, then, for the first approximation, we have $x = a + \frac{1}{b} = \frac{ab + 1}{b}$. Next, place $y = b + \frac{1}{y'}$. The transformed equation in y' must have its values greater than unity. Let c be the

entire part of the value of y'. Then, approximatively, $y = b + \frac{1}{c} = \frac{cb+1}{c}$, and, for a second approximation to x, we have, $x = a + \frac{1}{y} = a + \frac{c}{cb+1}$. To find a third approximate value for x, let $y' = c + \frac{1}{y''}$, and we may thus continue to approximate to the value of x until the result is as accurate as desired.

To apply these principles, let us take the equation, $x^3 - x - 5 = 0$. We see that 1 and 2 give contrary signs, hence, $x = 1 + \frac{1}{y}$. Substituting this value of x, we have $5y^3 - 2y^2 - 3y - 1 = 0$. The entire part of the value of y is 1, hence, for first approximation to x, we have $x = 1 + \frac{1}{y} = 1 + \frac{1}{1} = 2$, which is plainly too great. Next, making $y = 1 + \frac{1}{y'}$, we get $y'^3 - 8y'^2 - 13y' - 5 = 0$. In this, 9 and 10 give contrary signs. Then, $y = 1 + \frac{1}{9} = \frac{10}{9}$, and $x = 1 + \frac{9}{10} = \frac{19}{10}$, for the second approximation.

Now, make $y' = 9 + \frac{1}{y''}$, and the transformed equation in y'' will be, $41y''^3 - 86y''^2 - 19y'' - 1 = 0$; a change of sign between the results given by 2 and 3. Hence, $y' = 9\frac{1}{2} = \frac{19}{2}$, $y = 1 + \frac{2}{19} = \frac{21}{19}$, and $x = 1 + \frac{19}{21} = \frac{40}{21}$, for the third approximate value of x. We find on trial the third approximate value a little too great, and the second a little too small. Hence, the true value lies between $\frac{40}{21}$ and $\frac{19}{10}$, and either of these numbers differs from the true value by less than $\frac{1}{210}$.

486. Let us take the simple example, $x^2 - 2 = 0$. Place $x = 1 + \frac{1}{y}$, then, $y^2 - 2y - 1 = 0$. We find that 2 and 3 give contrary signs. Hence, for first approximation, $x = 1 + \frac{1}{2} = \frac{3}{2}$. Now, make $y = 2 + \frac{1}{y'}$, and the equation in y will become $y'^2 - 2y' - 1 = 0$; and, again, 2 is the entire part of the value. Hence, $y = 2 + \frac{1}{2} = \frac{5}{2}$, and $x = 1 + \frac{2}{5} = \frac{7}{5}$, for a second approximation. On trial, we find $\frac{3}{2}$ too great, and $\frac{7}{5}$ too small, and these differ from each other by a tenth; therefore, the true value differs from either by less than a tenth. For a third approximate value we would find, $x = \frac{17}{12}$, which is a little too great, but is within $\frac{1}{60}$ of the true value. The fourth approximate value, $\frac{41}{29}$, is too small, but differs from the true value by less than $\frac{1}{321}$. By continuing thus the process, we would find the approximations alternately too great and

too small, and thus can tell at every step how near we have come to the true values. We see that Lagrange's method enables us to determine the proximity to the true value at every stage of the work, and this is the chief advantage claimed for it over the process of Newton.

EXAMPLES.

1. $x^3 - 7x + 7 = 0$. *Ans.* $x = \frac{22}{13}$, after three approximations.

2. $x^4 - x^2 - 6 = 0$. *Ans.* $x = \frac{7}{4}$, after third approximation.

1st approximation, 2; 2d approximation, $\frac{5}{3}$; 3d approximation, $\frac{7}{4}$.

GENERAL SOLUTION OF NUMERICAL EQUATIONS.

487. When we have a numerical equation of any degree to solve, we must first find its rational values by the process of divisors, if it is under the proposed form. But, if it is not, we must transform it into another equation in y, so that the coefficient of the first term shall be plus unity, and the other coefficients entire. Then, knowing the relation between y and x, we can determine the values of x when those of y have been found. Next we must ascertain whether any of the values are repeated (Art. 475). Having thus found all the rational values, we next divide by the factors corresponding to them. Then, by means of the equation of the squares of the differences, we can ascertain what series of numbers to substitute in the reduced equation (Art. 481). The final step is to find, approximatively, the irrational values by either of the three processes explained. Now, if the number of rational and irrational values be subtracted from the degree of the given equation, the difference will be the number of imaginary values.

GENERAL EXAMPLES.

1. Find the three values of the equation, $2x^3 - \frac{152x^2}{5} + 106x - 20 = 0$.
Ans. $x = \frac{1}{5}$, 5, and 10.

2. Solve the equation, $x^5 - 11x^3 + 7x^2 + 28x - 28 = 0$.
Ans. $x = \pm 2$, 1·356, 1·692, and $-3{\cdot}044$.

3. Solve the equation, $x^3 + \frac{17x^2}{6} - \frac{2}{3}x - \frac{1}{2} = 0$.
Ans. $x = \frac{1}{2}$, $-\frac{1}{3}$, and -3.

4. Solve the equation, $x^4 - 10x^3 - 12x^2 + 92x + 280 = 0$.
Ans. $x = 10$, and $4{\cdot}302$.

5. Solve the equation, $2x^5 - 2x^4 - \frac{9x^3}{2} + \frac{9x^2}{2} + x - 1 = 0$.
Ans. $x = 1, \frac{1}{2}, -\frac{1}{2}$, and $\pm\sqrt{2}$.

6. Solve the equation, $x^7 - 4x^6 - 4x^5 + 39x^4 - 60x^3 - 28x^2 + 224x - 224 = 0$.
Ans. $x = 2, 2$, and $x = 2{\cdot}828$.

7. Solve the equation, $x^4 - 2x^3 - 31x^2 + 22x + 280 = 0$.
Ans. $x = 4$, and $5{\cdot}134$.

STURM'S THEOREM.

488. When we have, by means of the process of divisors and the method of equal values, detected all the rational values, and then, by the aid of the equation of differences, discovered all the irrational values, we can determine the number of imaginary values, by subtracting the number of rational and irrational values from the degree of the equation. When this difference is zero, there are, of course, no imaginary values.

But this method is circuitous, and is, moreover, rendered tedious by the employment of the equation of differences. *Sturm's Theorem determines directly the number of imaginary values, and dispenses with the equation of differences.*

Let $X = x^m + Px^{m-1} + Qx^{m-2} + \ldots\ldots + Tx + U = 0$ *be an equation cleared of its equal values.* Take the derivative of X, and call it X′; then, $X' = mx^{m-1} + (m-1)Px^{m-2} + (m-2)Qx^{m-3} + \ldots\ldots + T$. Now, divide X by X′, and continue the division until we get a remainder in x of a lower degree than X′. Call this remainder, with its sign changed, X″. Divide X′, in like manner, by X″, and continue the division until the remainder is a degree lower than the divisor. Change, in the same way, the signs of all the terms of this remainder, and call the resulting expression X‴. Pursue the same process until we get a remainder independent of x, which must be so eventually, since the given equation, by hypothesis, contains no equal values. This will be the $(m-1)^{\text{th}}$ remainder, and all the expressions, including X and X′, will make up $(m+1)$ functions. Hence, designate by X^{m+1}, the last remainder, with its sign changed. We will then have the following expressions, Q, Q′, &c., designating quotients:

$$X = X'Q - X'' \quad (A),$$
$$X' = X''Q' - X''' \quad (B),$$
$$X'' = X'''Q'' - X^{iv} \quad (C),$$
$$X''' = X^{iv}Q''' - X^{v} \quad (D),$$
$$X^{iv} = X^{v}Q^{iv} - X^{vi} \quad (E),$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$X^{m-1} = X^{m}Q^{m-1} - X^{m+1} \quad (W).$$

In preparing the dividends for division, we may, as in the process for finding the greatest common divisor, multiply by any *positive constant.* In like manner, we may suppress any *positive constant* common to all the terms of any of the successive remainders.

489. The theorem we have to demonstrate may be enunciated as follows:

If we substitute any number, m, for x in the above series of functions, X, X′, X″, &c., and count the variations of signs in the *results,* and then substitute any other number, as n, for x in the same series, and count again the variations of signs in the *results,* the *difference between the number of variations in the two cases will express exactly the number of real values between the numbers* m *and* n. If, then, m be taken as the superior limit of the positive values, and n as the superior limit of the negative values, the difference between the number of variations will express the total number of real values. When this difference is zero, there are no real values; when it is equal to the degree of the equation, all the values are real.

490. The demonstration of the theorem depends upon the four following principles:

1. *No two consecutive functions can vanish for the same value of* x.

For, if two functions, as X″ and X‴, could disappear together, equation (C) would give $0 = 0 - X^{iv}$. Hence, $X^{iv} = 0$, and X‴ and X^{iv} being zero, equation (D) would give $X^{v} = 0$. And so all the functions in succession would become zero, until we would finally have $X^{m+1} = 0$. But this would indicate a common divisor, and, consequently, equal values, which is contrary to the hypothesis with which we set out.

2. *When such a value is given to* x *as to make any function other than* X *equal to zero, the functions adjacent to the one disappearing will be affected with contrary signs.*

For example, let $X^{iv} = 0$. Then, equation (D) will give $X''' = -X^{v}$; and, since the own sign of X^{v} may be either plus or minus, this equation may be written $X''' = -(\pm X^{v})$. Now, if X^{v} be affected with the positive sign, the second member will be negative; and, since the signs of the two members of an equation must always be the same, X''' will be negative, and, therefore, of a contrary sign to X^{v}. But, if the own sign of X^{v} be negative, then the second member will be positive. Hence, X''' will be positive, and, therefore, of a contrary sign to X^{v}.

3. *The disappearance of any function other than* X *for a particular value of* x, *will neither increase nor decrease the variations of signs in the results of the series,* X, X', X'', *&c.*

Suppose that one of the intermediate functions, as X^{iv}, becomes zero, for the value $x = b$. Then, by the first principle, neither X''' nor X^{v} can be zero for $x = b$; and, by the second principle, they must have contrary signs. There is, then, one variation of sign between X''' and X^{v} when $x = b$. We are to show that this variation was not caused by the disappearance of X^{iv}. For, however little $x = b$ may differ from a value of X''' or X^{iv}, h may be taken smaller than this difference. And, moreover, since a quantity can only change its sign in passing through zero or infinity, it is plain that, between $x = b$ and $x = b - h$, neither X''' nor X^{v} can undergo any change of sign. They will still be affected with contrary signs, and, for $x = b - h$, we will have either $+X'''$, $\pm X^{iv}$, $-X^{v}$, or else, $-X'''$, $\pm X^{iv}$, $+X^{v}$. We affect X^{iv} with the ambiguous sign, since we do not know whether its sign is positive or negative. Now, read the double sign either way, and take either of the preceding expressions, and you will have one variation. Hence, before x arrives at the value b, which causes the disappearance of X^{iv}, the three successive functions give one variation. Now, when x passes beyond b, and becomes $b + h$ (h being less than the difference between b and a value of either X''', or X^{v}), X^{iv} will change its sign, but X''' and X^{v} will not. Hence, for $x = b + h$, we will have either $+X'''$, $\mp X^{iv}$, $-X^{v}$, or $-X'''$, $\mp X^{iv}$, $+X^{v}$. And, reading the ambiguous sign either way, in either of the above expressions, we will have one variation. Hence, both before and after x reached the value b, which made $X^{iv} = 0$, we find a variation of sign among the three functions. Then $x = b$, or the vanishing of the intermediate function, X^{iv}, has not caused that variation.

4. *Every passage of* X *through zero causes a loss of one variation when* x *is ascending towards the superior positive limit, or a gain of one variation throughout the series,* X, X′, *&c., when* x *is descending towards the superior negative limit.*

For, let $x = a$ be a value in the equation, $X = 0$, and take u less than the difference between a and a value of X′. Then, X will have a different sign, when $a + u$ is substituted for x, from what it would have were $a - u$ substituted for x. Hence, in the passage of x from $a - u$ to $a + u$, X will undergo a change of sign, but X′ will not. since, by hypothesis, no value of X′ is passed over. Neither will any intermediate function, X″, X‴, &c., undergo a change of sign unless it pass through zero, and, by the third principle, such passage will not affect the total number of variations. Therefore, in the passage of x from $a - u$ to $a + u$, either a permanence, existing between X and X′, is changed into a variation, or else a variation is changed into a permanence.

We will now examine the order in which this change takes place. Designate by X_1 what X becomes when $x = a - u$. Next, develop all the terms of X by Art. 456; we will have $X_1 = A - A'u + \frac{A''u^2}{1 \cdot 2} - \frac{A'''u^3}{1 \cdot 2 \cdot 3} + \&c.$, (R), in which A denotes what X becomes when $x = a$; A′ is the derivative of A, A″ the derivative of A′, &c. But, since a is a value of x, A must be equal to zero. Then, equation (R) may be written, $X_1 = -u\left(A' - \frac{A''u}{1 \cdot 2} + \frac{A'''u^2}{1 \cdot 2 \cdot 3} - \&c.\right)$ Now, u may be taken so small that the first term will be greater than the algebraic sum of all the other terms within the brackets. Hence, the sign of the quantities within the parenthesis will depend upon that of A′. If, therefore, the own sign of A′ is positive, the second member will be negative. Then, X_1 must be negative, and, therefore, of a contrary sign to A′. But, if the own sign of A′ be negative, the second member will be positive. Then, X_1 will be positive, and X_1 and A′ will be affected with contrary signs. And, since X_1 and A′ represent what X and X′ become when $a - u$ is substituted for x, we see that there is a variation of signs between these functions below a value, $x = u$, which satisfies the equation, $X = 0$. Now, suppose $x = a + u$, and, therefore, above a value in the equation, $X = 0$. Develop as before, designating by X_2 what X becomes when $a + u$ takes the place of x. Then, $X_2 = u\left(A' + \frac{A''u}{1 \cdot 2} + \frac{A'''u^2}{1 \cdot 2 \cdot 3} + \&c.\right)$, in which u is so small that the sign of A' controls that of the parenthesis. We see, that, whether the own sign of A' be positive or negative, X_2 must be affected with the same sign. Hence, when $x = a + u$, and, therefore, above a value, there is a permanence of sign between X and X′. Therefore, in the passage of x towards the superior positive limit, a variation is lost, and changed into a permanence between X and X′, whenever X becomes equal to zero; and, as the other functions neither gain nor lose variations, one variation only is lost throughout the entire series whenever $X = 0$.

If we had begun with $a + u$, and passed down to $a - u$, it is evident that one variation would have been gained, every time that x reached a value in the equation, $X = 0$.

So, then, whenever x, increasing or decreasing by insensible degrees, reaches a value which will satisfy the equation, $X = 0$, one variation is lost or gained in the signs of the series, X, X′, X″, &c. Hence, if we take any number, as m, and substitute it for x in all the functions, and count the number of variations in the signs of the results, and then take any other number, as n, and treat it in like manner, the difference between the number of variations in the two cases will express the number of real values lying in the given equation between m and n. Moreover, it is plain that if $m = +\infty$, and $n = -\infty$, this difference will express the *total* number of real values in the equation. This would be equally true if we used the superior positive limit and the superior negative limit. But, there is this advantage in the employment of $+\infty$, and $-\infty$; the substitution need only be made in the leading term of the successive functions, for then the sign of this term would control the signs of all the other terms in the same function.

491. Having determined the number of real values, the next step is to determine the initial figure of each one of these values. To do this, we substitute the natural numbers, 0, 1, 2, 3, &c., until we get the same number of variations of signs in the entire series as was given by $+\infty$. The number that gives the same variations, as $+\infty$, is the superior positive limit. Substitute, in like manner, 0, -1, -2, -3, &c., until we get the same number of variations as given by $-\infty$. We then have the superior negative limit. The least of the numbers, whether positive or negative, between which a variation is lost or gained, is the initial figure, or entire part of a real value. Should there be two or more variations lost or gained between consecu-

numbers, then there are as many values as there are changes of sign, which have the same initial figure.

We will illustrate by a simple example.

Take equation, $x^3 + 3x^2 - 1 = 0 = x$. Then its derivative, $X' = 3x^2 + 6x$.

Now, multiply all the terms of X by 9, and divide by X′. After two divisions, we will get a remainder of a lower degree than X′; this is $-2x - 9$. Change its sign, and call it X″. Then, $X'' = 2x + 9$. Multiply X′ by 4, and, after two divisions, by X″, we will have a remainder, $+ 207$. Hence, $X''' = -207$, and we will have

$$\begin{aligned} X &= x^3 + x^2 - 1 = 0 \\ X' &= 3x^2 + 2x \\ X'' &= 2x + 9 \\ X''' &= -207. \end{aligned}$$

Making $x = +\infty$, in the leading terms of these functions, the order of the signs will be

$$+ + + -,\ \text{one variation.}$$

And for $x = -\infty$, it will be

$$- + - -,\ \text{two variations.}$$

Hence, $2 - 1 = 1$, real value.

To find the initial figure of this value, make $x = 0, 1, 2, 3,$ &c., and write the signs of the results beneath their respective functions. We will have

	X,	X′,	X″,	X‴,
then, $x = 0$,	-1,	0,	$+9$,	-207, two variations.
$x = 1$,	$+1$,	$+5$,	$+11$,	-207, one variation.

Hence, 1 is the superior limit of the value sought, and 0 is its initial figure. The decimal part of the value can be found by the process for the irrational values.

The transformed equation in y, is $y^3 + y^2 - 1 = 0$; and that in z, $z^3 + 10z^2 - 1000 = 0$. And, since 7 and 8 give results with contrary signs, 7 is the tenths of the required value. Then, $x = 0{\cdot}7$ is an approximate value of x.

The approximation may be carried as far as desired by the method of irrational values.

We need not substitute 0, — 1, — 2, &c., in the equation, since there is but one real value, and that has been shown to be positive.

492. Whenever only one variation is lost between two consecutive numbers, Sturm's theorem affords no advantage over the process for finding the irrational values, except that of detecting directly the number of imaginary values. But, when there are two or more real values comprised between two consecutive numbers, we are enabled by means of the theorem to dispense with the equation of differences, which otherwise must be employed to find the decimal part of those values. We will now explain the chief advantage of the theorem.

Let us take the equation, $x^5 - 2x^3 - 2x^2 + 4 = 0$. We will have the following series of functions:

$$X = x^5 - 2x^3 - 2x^2 + 4 = 0,$$
$$X' = 5x^4 - 6x^2 - 4x,$$
$$X'' = 4x^3 + 6x^2 - 20,$$
$$X''' = -42x^2 - 168x + 300,$$
$$X^{iv} = -2880x + 3840,$$
$$X^{v} = -\text{, a constant.}$$

$+\infty$ gives $+ + + + - -$, one variation,

$-\infty$ gives $- + - + + -$, four variations.

Hence, there are three real values.

Proceeding as before, we have

		X,	X',	X'',	X''',	X^{iv},	X^{v}.	
When,	$x = 0$	+	0	—	0	+	—,	three variations,
"	$x = 1$	+	—	—	+	+	—,	three variations,
"	$x = 2$	+	+	+	+	—	—,	one variation.

Since + 2 gives the same number of variations as $+\infty$, it is the superior positive limit. Moreover, as there are two variations lost between 1 and 2, there are two real values between them. Had we substituted the natural numbers, 0, 1, 2, 3, &c., in the given equation, we would have found no change of sign, because an even number of values lay between consecutive numbers. To detect these values, without the aid of Sturm's theorem, we must either have recourse to the equation of differences, or to minute and tedious substitutions.

So, then, it is evident that whenever an equation contains two, four, six, or any number of even values between consecutive numbers, Sturm's theorem enables us to detect these values in the shortest possible manner.

493. But, in addition to this, the theorem gives us the means of finding the decimal part of the irrational values in a shorter and better manner than by the usual process, as we will now show.

Since 1 is the entire part of both the irrational values, the first step, according to the usual process for finding the irrational values, is to transform the equation in x into another in y, so that the values of y shall be less by unity than those of x.

The transformed equation in y is

$$y^5 + 5y^4 + 8y^3 + 2y^2 - 5y + 1 = 0,$$

and that in z, is

$$z^5 + 50z^4 + 800z^3 + 2000z^2 - 50000z + 100000 = 0.$$

We find, on trial, that 0, 1, and 2 give positive results; 3 and 4 give negative results; 5, and all numbers above 5, give, again, positive results. Hence, 2 and 4 are the tenths of the sought values; and 1·2, and 1·4 are those values, approximatively.

Now, if, in the transformed equation in z, two values had lain between consecutive numbers, we must have had recourse either to minute substitutions, or to the equation of differences. If, for instance, the second value had been 1·26 instead of 1·4, the tenths would have been the same for both values, and the substitution of the natural numbers in z would have given no change of sign.

To obviate a difficulty that might occur in more than one of the transformed equations, we proceed thus,

We take all the functions, X, X′, X″, &c., and transform them into others in y, so that the values of y shall differ from those of x, by the initial figure, which, in this case, is unity. We will then have the series,

$$Y = y^5 + 5y^4 + 8y^3 + 2y^2 - 5y + 1,$$
$$Y' = 5y^4 + 20y^3 + 24y^2 + 4y - 5,$$
$$Y'' = 4y^3 + 18y^2 + 24y - 10,$$
$$Y''' = -42y^2 - 252y + 90,$$
$$Y^{iv} = -2880y + 960,$$
$$Y^{v} = -\text{, a constant.}$$

After having found Y, as indicated, we may get its derivative, Y′, and then divide Y by Y′, and proceed as we did when getting the functions, X, X′, X″, &c. But, in general, the better method is to transform the functions in x into others in y, so that the values of y shall be less than those of x, by the initial figure of the sought value.

The transformed functions in z are:

$$Z = z^5 + 50z^4 + 800z^3 + 2000z^2 - 50000z + 100000,$$
$$Z' = 5z^4 + 200z^3 + 2400z^2 + 4000z - 50000,$$
$$Z'' = 4z^3 + 180z^2 + 2400z - 10000,$$
$$Z''' = -42z^2 - 2520z + 9000,$$
$$Z^{iv} = -2880z + 9600,$$
$$Z^{v} = -, \text{ a constant.}$$

And we have the following results:

	Z,	Z′,	Z″,	Z‴	Z^iv,	Z^v.	
When $z = 0$	+	—	—	+	+	—,	three variations,
" $z = 1$	+	—	—	+	+	—,	" "
" $z = 2$	+	—	—	+	+	—,	" "
" $z = 3$	—	—	—	+	+	—,	two variations,
" $z = 4$	—	—	+	—	—	—,	" "
" $z = 5$	+	+	+	—	—	—,	one variation.

We need go no further in the substitution, since 5 gives the same number of variations as $+ \infty$. We see that there is a variation lost between 2 and 3, and another between 4 and 5. Hence, 2 and 4 are the tenths in the required values, as we before found. Now, if there had been two, or any number of values having the same initial decimal figure, 3, for example, there would have been as many variations lost or gained as there were values having the same initial decimal.

494. We see, then, the two great advantages of Sturm's theorem in finding the irrational values over the other three processes described: 1st. When an equation comprises an even number of values between two consecutive numbers, it enables us to detect these values without minute substitutions, or the employment of the equation of differences. 2d. When two or more values have the same initial decimal figure, it enables us to tell the exact number of those values. If we add to these two advantages the one first mentioned, that of detecting *directly* the

number of real, and, consequently, the number of imaginary values, we can see how important the theorem is.

To get the hundredths, the functions in z may be transformed into others in s, so that the values of s shall differ from those of z by the initial decimal figures; in this case, 2 and 4. Then, again, transform the functions in s into others in w, so that the values of w shall be ten times greater than those of s. We will then have a series of functions, W, W′, W″, W‴, &c., in which we may substitute the natural numbers, 0, 1, 2, &c., until we get as many variations as $+\infty$ gives in the series. The least of the consecutive numbers, between which a loss of variation occurs, will be hundredths in the sought value. We, of course, will have two series of functions in w, the one corresponding to 2, as the initial figure of decimals, and the other to 4.

495. Since there were three real values in the equation, $x^5 - 2x^3 - 2x^2 + 4 = 0$, and we have found but two positive values, the other must be negative. To determine this negative value, let us resume the functions,

$$X = x^5 - 2x^3 - 2x^2 + 4 = 0,$$
$$X' = 5x^4 - 6x^2 - 4x,$$
$$X'' = 4x^3 + 6x^2 - 20,$$
$$X''' = -42x^2 - 168x + 300,$$
$$X^{iv} = -2880x + 3840,$$
$$X^{v} = -\text{, a constant.}$$

Substituting 0, — 1, — 2, — 3, &c., in the foregoing series, we will have

		X	X′	X″	X‴	X^{iv}	X^{v}	
When,	$x = 0$	+	0	—	+	+	—,	three variations,
"	$x = -1$	+	+	—	+	+	—,	three variations,
"	$x = -2$	—	+	—	+	+	—,	four variations.

And, since — 2 gives the same number of variations as $-\infty$, it is the superior negative limit. And, since a gain of variation occurs between — 1 and — 2, the entire part of the negative value is — 1. In this case, we know that there is but one negative value; we may then proceed at once to find the decimal part of this value by the usual process for irrational values, without forming the functions, Y, Y′, Y″, &c.

The transformed equation in y is

$$y^5 - 5y^4 + 8y^3 - 6y^2 + 3y + 3 = 0,$$

and that in z,

$$z^5 - 50z^4 + 800z^3 - 6000z^2 + 30000z + 300000 = 0.$$

A change occurs between 4 and 5. Hence, 4 is the tenths of the sought value, and we have $x = -1{\cdot}4$ for the approximate value. Now, diminish the values in the equation in z by 4, and the transformed equation in s will be

$$s^5 - 20s^4 + 1760s^3 - 21040s^2 + 130480s + 21536 = 0,$$

and that in w will be,

$$w^5 - 200w^4 + 176000w^3 - 21040000w^2 + 1304800000w + 2153600000 = 0.$$

A change of sign occurs between the results after the substitution of -1 and -2. Hence, 1 is the hundredths of the negative value. And we have, for a second approximation, $x = -1{\cdot}41$.

Remark.

496. There is one point of considerable importance in the demonstration, which deserves to be attended to. It has been shown that a variation is lost or gained between X and X′, every time that X becomes equal to zero, or that x passes a value. It might be asked, then, Why not confine the substitutions to X and X′?

It is to be observed, that a change of variation only takes place between X and X′ when very minute substitutions are made. For, in the demonstration of the fourth principle, we supposed u to be indefinitely small. Now, if we substitute a number, p, which will make the signs of X and X′ contrary, and again substitute another, p', there being two values of X, and none of X′, between p and p', then X and X′ will still be affected with contrary signs. So, that, between p and p', there will be no change of variation, though there are two real values. But, if one value of X′ lies between p and p', there will be one variation lost, and but one. For, whilst X changes its sign twice, X′ will change its sign once.

Take $\quad X = 6x^2 - 5x + 1$; then, $X' = 12x - 5$.

There is a variation between X and X′ when $x = 0$, and this is changed into a permanence when $x = 1$. There are two values of X between 0 and 1, and one of X′.

Again, if there were three, five, or any odd number of values of X passed over, and none of X′, there would be one variation lost or gained, and but one. But, if, at the same time, an odd number of values of X′ were passed over, there would be no change of variation. It is plain, then, that the loss and gain of variation between X and X′ will only correspond to the number of real values, when the substitutions are so minute that no value of either X or X′ is contained between them.

Scholium.

497. Whenever any function is constantly positive for all values of x, we need not form any other functions, but only count the number of variations given by $+\infty$, and $-\infty$, in the constantly positive function, and in the functions which precede it. For, if we formed the succeeding functions, they would give the same number of variations for $+\infty$ that they would for $-\infty$. Hence, the difference between the number of variations of these functions must always be zero. To show this, we take for granted that a function, which always remains positive for all values of x, must be of an even degree, since it will contain only imaginary values. The next succeeding function may have its leading term either positive or negative, but the next must be negative, since any value that reduces the function, consecutive with the positive one to zero, must cause the adjacent functions to be affected with contrary signs. All the functions of an odd degree may be either positive or negative, but those of an even degree must be alternately positive and negative.

Let $x^6 + mx^5 + \&c.$, be the constantly positive function. We will then have the series,

$$\begin{aligned}
&+x^6 + mx^5 + \&c. &&= X^r,\\
&\pm nx^5 + \&c. &&= X^{r+1},\\
&-px^4 - \&c. &&= X^{r+2},\\
&\pm rx^3 + \&c. &&= X^{r+3},\\
&+sx^2 + \&c. &&= X^{r+4},\\
&\pm tx \;\; + \&c. &&= X^{r+5},\\
&-A, \text{ a constant}, &&= X^{r+6},
\end{aligned}$$

Now, suppose the ambiguous sign to be plus throughout, we will have for $+\infty$, these results,

$$+ + - + + + -, \text{ 3 variations,}$$

and for $-\infty$, these results,

$$+ - - - + - -, \text{ 3 variations.}$$

Next, suppose the ambiguous sign to be minus throughout, then, $-\infty$ will give

$$+ + - + + + -, \text{ 3 variations,}$$

and $+\infty$ will give

$$+ - - - + - -, \text{ 3 variations.}$$

We will find like results when some of the ambiguous signs are taken as positive, and the rest as negative.

GENERAL EXAMPLES.

1. Given, $x^3 - x^2 - 7 = 0$, to find x.

$$X = x^3 - x^2 - 7 = 0,$$
$$X' = 3x^2 - 2x,$$
$$X'' = 2x + 63,$$
$$X''' = -\text{, a constant.}$$

$$+\infty \text{ gives } + + + -, \text{ 1 variation,}$$
$$-\infty \text{ gives } - + + -, \text{ 2 variations.}$$

Hence, one real value, and $x = 2{\cdot}310$.

2. Find one value of x in the equation, $x^4 - 2x^3 + x^2 - 5 = 0$.

Ans. $x = 2{\cdot}076$.

$$X = x^4 - 2x^3 + x^2 - 5 = 0,$$
$$X' = 4x^3 - 6x^2 + 2x,$$
$$X'' = x^2 - x + 20,$$
$$X''' = 2x - 1,$$
$$X^{\text{iv}} = -A\text{, a constant.}$$

$$+\infty \text{ gives } + + + + -, \text{ 1 variation,}$$
$$-\infty \text{ gives } + - + - -, \text{ 3 variations.}$$

Hence, two real values.

3. Given, $x^3 - x - 7 = 0$, to find x. *Ans.* $x = 2{\cdot}086$.

$$X = x^3 - x - 7 = 0,$$
$$X' = 3x^2 - 1$$
$$X'' = 2x + 21,$$
$$X''' = -A.$$

$+\infty$ gives $+\ +\ +\ -$, 1 variation,
$-\infty$ gives $-\ +\ -\ -$, 2 variations.

Hence, one real value.

4. Given, $x^3 - x^2 + 7 = 0$, to find x. *Ans.* $x = -1{\cdot}63$.

$$X = x^3 - x^2 + 7 = 0,$$
$$X' = 3x^2 - 2x,$$
$$X'' = 2x - 63,$$
$$X''' = -A.$$

$+\infty$ gives $+\ +\ +\ -$, 1 variation,
$-\infty$ gives $-\ +\ -\ -$, 2 variations.

Hence, one real value.

DESCARTES' RULE.

498. *An equation cannot have a greater number of negative values than there are permanences of sign from + to +, or from — to —; nor can it have a greater number of positive values than there are variations of sign from + to —, or from — to +.*

When the adjacent terms of an equation are affected with the same sign, a *permanence* is said to exist between them; and when they are affected with contrary signs, there is a *variation* between them.

In applying Descartes' rule to an equation, every sign is read twice, except the first and last. Thus, in the equation, $x^4 - 4x^3 + 12x^2 + x - 5 = 0$, there is a *variation* between the first and second terms, a *variation* between the second and third terms, a *permanence* between the third and fourth terms, and a *variation* between the fourth and fifth terms. In all, there are three variations and one permanence.

When a term is missing from an equation, it must be supplied with the coefficient, ± 0. Then, in reading the signs, we must first count the variations and permanences, regarding the ambiguous sign as positive, and then count again, regarding the ambiguous sign as minus. Thus, taking the equation, $x^2 - 4 = 0$, we must write it $x^2 \pm 0x - 4 = 0$. Counting the upper sign, we have a permanence between the first and second terms, and a variation between the second and third terms. Counting the lower sign, we have a variation between the first and second terms, and a permanence between the second and third terms. In whatever way, then, we read the ambiguous sign, there will be one permanence, and one variation.

If there are any number of missing terms, they must be supplied in like manner with positive or negative zero coefficients.

499. To demonstrate the rule of Descartes, it is necessary to show that the multiplication of any equation by a factor, corresponding to a negative value, will introduce into the new equation *at least* one more *permanence* than existed in the old; and that the multiplication by a factor, corresponding to a positive value, will introduce *at least* one more variation.

1st. Let us take the equation, $x^m - Px^{m-1} + Qx^{m-2} + Rx^{m-3} - Sx^{m-4} \ldots + Tx - U = 0$, and multiply by a factor, $x + a$, corresponding to a negative value. The resulting equation will be

$$\begin{array}{r|r|r|r|r|r|l} x^{m+1}-P & x^m+Q & x^{m-1}+R & x^{m-2}-S & x^{m-3}\ldots+T & x^2-U & x=0 \\ +a & -Pa & +Qa & +Ra & -Sa & +Ta & -Ua \end{array} \quad (M)$$

Now, it is plain that, if we suppose the coefficients in the upper column are, throughout, greater than the corresponding coefficients in the lower column, there will be the same number of permanences in the new as in the old equation, until we get to the last two coefficients (the coefficients of x and x^0), we will then have one more permanence than in the given equation. And, if we suppose the coefficients of the lower column to be greater throughout, there will be a new permanence between the first and second terms, and the same succession of signs in the remaining terms. Moreover, it is evident, that if the lower column sometimes prevailed, and sometimes did not, there might be more than one permanence introduced. For instance, if a were greater than P, Pa less than Q, Ra greater than S, T greater than Sa, Ta greater than U, there would be but one variation in the new equation.

It is even possible to change all the variations into permanences, by multiplication by a factor corresponding to a negative value. As an illustration, take the equation, $x^6 - 2x^5 + 12x^4 - 8x^3 + 36x^2 - x + 15 = 0$, and multiply by $x + 4$. The new equation will be,

$$\begin{array}{r|r|r|r|r|r|r} x^7 - 2 & x^6 + 12 & x^5 - \ \ 8 & x^4 + 36 & x^3 - \ \ \ 1 & x^2 + 15 & x = \ \ 0 \\ +4 & -\ \ 8 & +48 & -32 & +144 & -\ \ 4 & +60' \end{array}$$

or, $\quad x^7 + 2x^6 + 4x^5 + 40x^4 + 4x^3 + 143x^2 + 11x + 60 = 0.$

By this example we see that an equation, containing only variations, is changed into another containing only permanences, by multiplying the former by a factor corresponding to a negative value. Seven permanences have been introduced where none existed before. And, by recurrence to equation (M), we see that it is impossible to read the signs in any order, without having one more permanence than in the given equation. Now, the given equation may have been one of the first degree, m being equal to one, and P, Q, R, S, and T being equal to zero. Then, if the sign of U were positive, there would be one negative value, and conversely. The new equation (after multiplication by $x + a$) would be of the second degree, and would contain, at least, one more permanence than the old. And, by multiplying this new equation by another factor corresponding to a negative value, we would introduce, at least, one more permanence. And so, by continuing the process, it could be shown that, whatever might be the degree of the equation, the number of permanences must always be equal to, or exceed the number of negative values.

500. 2d. By a similar course of reasoning, we could show that the the multiplication by a factor corresponding to a positive value would introduce, at least, one *variation ;* or, in other words, that the number of positive values can never exceed the number of variations.

Take, as an illustration, the equation,

$$x^6 - 4x^5 + 12x^4 + 8x^3 + 30x^2 - x + 15 = 0,$$

and, multiply it by the factor, $x - 2$, the resulting equation will be,

$$x^7 - 6x^6 + 20x^5 - 16x^4 + 14x^3 - 61x^2 + 17x - 30 = 0,$$

and has gained three variations.

501. It is plain that the preceding reasoning has been on the supposition that a was a real value, otherwise we could not, in equation

(M), have instituted any comparisons between a and $-\mathrm{P}$, $-\mathrm{P}a$, and $+\mathrm{Q}$, $+\mathrm{R}a$, and $-\mathrm{S}$, &c. In case, then, that there are imaginary values in an equation, the rule of Descartes only points out limits, beyond which the positive and negative values cannot go. But, when the equation contains only real values, the number of positive values will be exactly equal to the number of variations, and the number of negative values exactly equal to the number of permanences. To show this, let $m =$ degree of the equation, $n =$ number of real negative values, $p =$ number of real positive values, $b =$ number of variations, $b' =$ number of permanences. A slight inspection will show that, in case of real values, $b + b' = m$; and we know that $n + p = m$. Hence, $b + b' = n + p$. But, since n cannot exceed b', and p cannot exceed b, we must have $n = b'$. For, if $n < b'$, then necessarily $p > b$, which cannot be. In like manner, we could show that we must have $p = b$.

Corollary.

502. In case of there being one or more missing terms in an equation, the rule of Descartes will enable us to detect imaginary values. We have only to supply the missing terms with plus or minus zero coefficients, count the variations and permanences when the upper sign is taken, and then again, when the lower sign is taken. If there be any discrepancy in the results, there will be imaginary values. Thus, take $x^2 + 4 = 0$; supplying the missing term, we have $x^2 \pm 0x + 4 = 0$. The upper sign taken in connection with the other two, gives two permanences, whilst the lower gives two variations. These discrepant results indicate imaginary values.

Take $x^4 + 3x^2 - 4 = 0$, then, $x^4 \pm 0x^3 + 3x^2 \pm 0x - 4 = 0$.

In one case, we have one variation and three permanences; in the other, three variations and one permanence. The difference in the two readings again indicates imaginary values.

GENERAL EXAMPLES.

1. What are the values in the equation, $x^3 - 2x^2 - 7x + 1 = 0$?

2. What are the values in the equation, $x^4 - 7x + 1 = 0$?

3. What are the values in the equation, $x^7 - 6x^6 + 7x^5 - 8x^4 + 9x^3 + 10x^2 - 11x + 12 = 0$?

4. What are the values in the equation, $x^5 - 1 = 0$?

5. What are the values in the equation, $x^3 - 1 = 0$?

6. What are the values in the equation, $x^5 + 1 = 0$?

7. What are the values in the equation, $x^5 + 4x^3 + 6x^2 + 4x + 1 = 0$?

ELIMINATION BETWEEN TWO EQUATIONS OF ANY DEGREE.

503. When one quantity depends upon another for its value, it is said to be a function of the quantity upon which it depends. Thus, in the equation, $y = 2x - 4$, y is a function of x, because every change in the value of x will produce a corresponding change in the value of y.

The mathematical symbol to designate a function may be F, or f, or the Greek letter, ϕ. Thus, to indicate that y is a function of x, we may employ the notation, $y = \mathrm{F}(x)$, or $y = f(x)$, or $y = \phi(x)$. The second member may contain *constants*, as well as the *variable*, x. Thus, y is a function of x in the foregoing equation, $y = 2x - 4$, the constants being 2 and -4.

504. The most general form of an equation of the m^{th} degree between two variables, x and y, is,

$$x^m + \mathrm{B}x^{m-1} + \mathrm{C}x^{m-2} + \mathrm{D}x^{m-3} + \mathrm{E}x^{m-4} \ldots\ldots + \mathrm{U} = 0;$$

in which B, C, D, &c., are functions of y.

B is supposed to be of the first degree in y, and of the form, $a + by$.

C is of the second degree in y, and of the form, $c + dy + ey^2$.

D is of the third degree in y, and of the form, $f + gy + hy^2 + ly^3$.

E is of the fourth degree in y, &c. &c.

U is of the m^{th} degree in y, and of the form, $u + my + \ldots\ldots + y^m$, and it does not contain x.

505. The equation is said to be *complete* when x enters into all the terms but the last, y into all the terms but the first, and when, also, the sum of the exponents of x and y in each term is equal to m.

506. Elimination between equations of a degree higher than the first is usually effected by means of the greatest common divisor. This method of elimination has already been explained (Art. 214), but we propose to demonstrate the process more rigorously, in two ways.

1. Let $A = 0$, and $B = 0$, be the proposed equations containing both x and y.

Now, if we knew beforehand a value m of y, that was common to the two equations, $A = 0$, and $B = 0$, and substituted this value in them, the new equations, $A' = 0$, and $B' = 0$, would contain only x and constants. Now, if n be a value of x, in the equation, $A' = 0$, its first member must be divisible by $x - n$, and it is plain that the given equations would not be simultaneous unless n would also satisfy the equation, $B' = 0$ (Art. 207). Hence, $x - n$ must also be a divisor of the equation, $B' = 0$. We see, then, that the hypothesis of a common value in y results in the condition of a common divisor in x. Conversely, if we can force the given equations to have a common divisor in x, they must have a common value in y. It is upon this principle that we seek for a common divisor between the first members of the equations, $A = 0$, and $B = 0$, and continue the process until we get a remainder freed from x. It is plain that, if we place this remainder equal to zero, and substitute the value of y, found from it in the last divisor, it will be an *exact* divisor, and will be the one sought.

From the foregoing reasoning it is evident, that if it be absurd to place the remainder, freed from x, equal to zero, the given equations are not simultaneous.

507. 2d. Let the successive quotients, in the process of dividing A by B, B by the remainder, &c., be designated by Q, Q′, &c., and the successive remainders by R, R′, &c. Then we will have.

$$A = BQ + R, \qquad (M)$$
$$B = RQ' + R', \qquad (N)$$
$$R = R'Q'' + R'', \qquad (O)$$
$$\&c. \qquad \&c.$$

Now, since, by hypothesis, $A = 0$, and $B = 0$, equation (M) will give $R = 0$. And, since $B = 0$, and $R = 0$, equation (N) will give $R' = 0$. From this we see that we have a right to equate, with zero, that remainder which is freed from x, and contains only y.

508. The above series of equations show, moreover, that if the remainder, which is freed from x, and the preceding divisor be placed

equal to zero, the system of values so found will satisfy the given equations. For, if R'' be that remainder, and R' the preceding divisor, when R'', and R' are equated with zero, equation (O) shows that R also $= 0$. And R and R' being equal to zero, from (N) we get, $B = 0$. And, since $B = 0$, and $R = 0$, equation (M) shows that we will also have $A = 0$.

Hence, the values of y, found by placing the last remainder equal to zero, may be substituted in the preceding divisor, in order to deduce the corresponding values of x. The importance of this remark consists in the fact, that the preceding divisor is of a lower degree than either of the original equations, and, therefore, more readily solved.

509. The reasoning in Art. 507 proceeds upon the supposition that the successive quotients, Q, Q', Q'', are all finite, and then, of course, the successive products, BQ, RQ', $R'Q''$, &c., will all be zero, when $B = 0$, $R = 0$, $R' = 0$, &c. But, if any of these quotients be fractional in form, it may happen that the value of y, found from placing the last remainder equal to zero, will reduce the denominator of the preceding divisor to zero also. In that case, we would have the product of zero by infinity, which is indeterminate.* Suppose, for example, $R = y - 1$, $Q = \frac{4}{y^2 - y}$. Then, $A = BQ + R$ becomes $A = B\left(\frac{4}{y^2 - y}\right) + y - 1$; or, when $R = 0$, $A = B\frac{(4)}{0} = B\,\infty$; or (since A and B are zero), $0 = 0\,\infty$, which may, or may not be, a true equation.

510. To avoid the fractional form of quotient, it may sometimes be necessary to multiply the dividend by some function of y, though this multiplication may possibly introduce some *foreign* values into the equation, as will be shown more fully hereafter.

511. We will now illustrate the foregoing principles by a few examples.

* Let $0\,\infty = A$, dividing both members by 0, we get $\infty = \frac{A}{0} = \infty$, a true equation. And this will evidently be true when $A = 1, 5, 10, 20$, or anything whatever.

Let $x^2 + y^2 - 8 = 0$, and $2x - 3y + 2 = 0$.

$$\begin{array}{lll} & \quad 4 & \\ & 4x^2+4y^2-32 & \mid \ 2x - 3y + 2 \\ & 4x^2+6xy+4x & \ \ 2x + (3y - 2) = \text{Quotient.} \\ \text{1st Remainder} = & 2(3y-2)x+4y^2-32 & \\ & 2(3y-2)x-9y^2+12y-4 & \\ \text{2d Remainder} = & \quad 13y^2-12y-28=0. & \end{array}$$

From which we get $y' = 2$, and $y'' = -\frac{14}{13}$; and these, when substituted in either of the given equations, give $x' = 2$, and $x'' = -\frac{34}{13}$.

Let $x^3 - y^3 - 7 = 0$, and $x - y - 1 = = 0$. Combining, we will get $x^2 + (y + 1)x + (y^2 + y) + (y + 1)$ for a quotient, and $y^2 + y - 2$ for a remainder.

The equation, formed by placing the remainder, freed from x, equal to zero, is called the *final equation.* In this example, the final equation, $y^2 + y - 2 = 0$, gives $y' = 1$, $y'' = -2$. And these values for y, when substituted, give $x' = 2$, and $x'' = -1$.

Let $x^3 - 3yx^2 + 3y^2x - 5x^2 + 10yx + 6x - y^3 - 5y^2 - 6y = 0$, and $x^3 - 5yx^2 + 8y^2x - x - 4y^3 + y = 0$. Then the first quotient is $+ 1$, and first remainder $(2y - 5)x^2 - 5y^2x + 7x + 10yx + 3y^3 - 5y^2 - 7y$. Preparing the last divisor for division by multiplying by $(2y - 5)^2$, we have

$$\begin{array}{l} x^2-5yx^2+8y^2x-x-4y^3+y \\ \qquad\qquad (2y-5)^2 \\ \hline (2y-5)^2x^3-20y^3x^2+32y^4x+196y^2x+10y^2x^2-125yx^2-160y^3x+20yx-25x-16y^5+80y^4-96y^3-20y^2+25y, \end{array}$$

and this, when divided by $(2y - 5)x^2 - 5y^2x + 7x + 10yx + 3y^3 - 5y^2$, gives as a quotient, $(2y - 5)x - 5y^2 + 15y - 7$, and as a remainder $y^4x - 10y^3x + 35y^2x - 50yx + 24x - y^5 + 10y^4 - 35y^3 + 50y^2 - 24y$. And, by factoring this remainder, we get $(y^4 - 10y^3 + 35y^2 - 50y + 24)x - y(y^4 - 10y^3 + 35y^2 - 50y + 24)$; or, $(x - y)(y^4 - 10y^3 + 35y^2 - 50y + 24)$. Rejecting the factor, $x - y$, as leading to arbitrary values, we have the final equation, $y^4 - 10y^3 + 35y^2 - 50y + 24 = 0$. This equation, when solved by the process of divisors, gives $y' = 1$, $y'' = 2$, $y''' = 3$, and $y^{\text{iv}} = 4$. And these, when substituted, give $x' = 3$, $x'' = 5$, $x''' = 5$, $x^{\text{iv}} = 7$.

512. Two things are suggested by this example. 1st. May not the multiplication by the factor, $(2y - 5)^2$, involving one of the unknown quantities, have introduced *foreign* values; that is, values which did not enter the given equations? 2d. $x - y$ being a common factor to

the remainder, is found to be also common to both the given equations. How are such factors to be treated? We will examine these subjects separately.

513. 1st. In regard to foreign values, we have this simple test. Place the multiplier equal to zero, find the value for y, and, from either of the given equations, the corresponding values of x; if these values be the same as some of those found from the final equation for y, with the corresponding values of x, then the system of common values in both x and y must be rejected. In the example, placing $(2y-5)^2 = 0$, we get $y = \frac{5}{2}$, a value diffeıent from those before found. Hence, the multiplier has not introduced a foreign value. But, if the multiplier, placed equal to zero, had given us, for example, $y = 1$, with the corresponding $x = 3$, then this system of values must be rejected.

514. How are factors to be treated which are common to both of the first members of the given equations? There may be three cases, but all lead to arbitrary values. 1st. The common factor may be a function of x only, $f(x)$. 2d. It may be a function of y only, $f(y)$. 3d. It may be a function of both x and y, $f(x, y)$.

When the common factor is $f(x)$ only, x will have determinate values, and y indeterminate. For, by placing $f(x) = 0$, we will get true values for x; but, when $f(x) = 0$, both equations will be satisfied, whatever values y may have. Take the equations, $(a + bx)(x^2y + 2y^2x - ay) = 0$, and $(a + bx)(4x^2y^3 - 2yx + my^2) = 0$. Placing $a + bx = 0$, we get $x = -\frac{b}{a}$, and both equations will be satisfied when $a + bx = 0$, whatever may be the values of y. Hence, $f(x) = 0$ gives x determinate, and y indeterminate. In like manner, a common factor, $f(y)$, would give y determinate, and x indeterminate. In the the third case, the common factor, $f(x, y) = 0$, will satisfy both equations; but, since we have a single equation, $f(x, y) = 0$, containing two unknown quantities, the values of both x and y must be indeterminate. Thus, take the equations, $(2x - 4y)(x^2 + 7y - 5) = 0$, and $(2x - 4y)(xy - 7x^2 + 5x^3 - 7y^2) = 0$. It is plain that both equations will be satisfied when $2x - 4y = 0$; but the equation, $2x - 4y = 0$, will give indeterminate values for both x and y.

Hence, we conclude that, when the given equations contain a common factor, it must be divided out. For, a common factor, $f(x)$, would give determinate values for x, but indeterminate for y; a common factor, $f(y)$, would give determinate values for y, and indeterminate for x; and a common factor, $f(x, y)$, would give x and y, both indeterminate.

515. Take the equations, $(x-1)(x^2-2xy+x-2)=0$, and $(x^2-xy-2)(x-1)=0$. Suppressing $x-1$, we have $x^2-2xy+x-2=0$, and $x^2-xy-2=0$, from which we get $y=1$, and $x=2$.

Take the equations, $(2y-6)(x^2y^2+5xy-y+1)=0$, and $(2y-6)(xy+4y-4)=0$. Suppressing $2y-6$, and combining the resulting equations, we get $16y^2-53y+37=0$ for the final equation. From which, $y'=1$, $y''=\frac{37}{16}$, $x'=0$, $x''=-\frac{84}{37}$.

Take the equations, $(2x-7y)(xy-y+5x^2-5x)=0$, and $(2x-7y)(xy+7x-7-y)=0$. Removing the common factor, we get, after combination, $5x^2-12+7=0$ for the final equation. From which, $x'=\frac{7}{5}$, $x''=1$, and, by substitution, $y'=-7$, and $y''=\frac{0}{0}$.

516. When the first members of the given equations can be resolved into factors of the first degree, or of a low degree, the elimination will be greatly facilitated.

Take the equations, $(x-1)(yx-3)(x^2-2xy)=0$, and $(yx-2x)(x^2-2y)=0$.

These equations can, obviously, be satisfied when

$$\left.\begin{array}{r} x-1=0 \\ \text{and } yx-2x=0 \end{array}\right|(A)\quad \left.\begin{array}{r} \text{when } x-1=0 \\ \text{and } x^2-2y=0 \end{array}\right|(B)\quad \left.\begin{array}{r} \text{when } yx-3=0 \\ \text{and } yx-2x=0 \end{array}\right|(C).$$

$$\left.\begin{array}{r} \text{when } yx-3=0 \\ \text{and } x^2-2y=0 \end{array}\right|(D)\quad \left.\begin{array}{r} \text{when } x^2-2xy=0 \\ \text{and } x^2-2y=0 \end{array}\right|(E)\quad \left.\begin{array}{r} \text{when } x^2-2xy=0 \\ \text{and } yx-2x=0 \end{array}\right|(F).$$

From (A) we get the system of values, $x=1$, and $y=2$.
From (B) " " " $x=1$, and $y=\frac{1}{2}$.
From (C) " " " $x=\frac{3}{2}$, and $y=2$.
From (D) " " " $x=\sqrt[3]{6}$, and $x=\frac{3}{\sqrt[3]{6}}$.
From (E) " " " $x=1$, and $y=\frac{0}{0}$.
From (F) " " " $x'=0$, $x''=4$, and $y'=\frac{0}{0}$, $y''=2$.

An artifice will sometimes enable us to decompose the first members of the given equations into their respective factors.

Take the equation, $x^2 - 2yx + 6x + y^2 - 6y + 5 = 0$, (A), and $x^2 + 2yx + y^2 + 6x + 6y + 5 = 0$, (B). From (A), we get $x^2 - 2yx + y^2 + 6x - 6y + 5 = 0$, or $(x - y)^2 + 6(x - y) + 9 - 4 = 0$; or (since the first three terms constitute a perfect square), $(x-y+3)^2 - 4 = (x - y + 3)^2 - (2)^2 = (x - y + 5)(x - y + 1)$ (Art. 50). From (B), we get $(x + y)^2 + 6(x + y) + 9 - 4 = 0$, or $(x + y + 3)^2 - 4 = (x+y+3)^2 - (2)^2 = (x+y+5)(x+y+1)$ (Art. 50). Hence, we have $(x - y + 5)(x - y + 1) = 0$, (A′), and $(x + y + 5)(x + y + 1) = 0$, (B′). And (A′) and (B′) will evidently be satisfied,

when $\left.\begin{array}{l} x-y+5=0 \\ x+y+5=0 \end{array}\right|$ (M) when $\left.\begin{array}{l} x-y+5=0 \\ x+y+1=0 \end{array}\right|$ (N) when $\left.\begin{array}{l} x-y+1=0 \\ x+y+5=0 \end{array}\right|$ (P),

and when $\left.\begin{array}{l} x - y + 1 = 0 \\ x + y + 1 = 0 \end{array}\right|$ (Q).

From (M) we get the system of values, $x = -5$, and $y = 0$.

From (N) " " " $x = -3$, and $y = +2$.

From (P) " " " $x = -3$, and $y = -2$.

From (Q) " " " $x = -1$, and $y = 0$.

Take the equations, $x^2 - 3yx - 3x + 2y^2 + 7y - 4 = 0$, (A).

and $$x^2 - 4x + xy - 4y = 0, \text{ (B).}$$

From (A) we get (by making $2y^2 = \frac{9y^2}{4} - \frac{y^2}{4}$, and adding and subtracting $\frac{9}{4}$),

$$\left(x - \frac{3y}{2}\right)^2 - 3\left(x - \frac{3y}{2}\right) + \frac{9}{4} - \left(\frac{y^2}{4} - \frac{5y}{2} + \frac{25}{4}\right) = 0$$

or, $\left(x - \frac{3y}{2} - \frac{3}{2}\right)^2 - \left(\frac{y}{2} - \frac{5}{2}\right)^2 = (x-y-4)(x-2y+1)$ (Art. 50).

From (B), we get $x(x - 4) + y(x - 4) = (x + y)(x - 4)$.

Hence, we have $(x - y - 4)(x - 2y + 1) = 0$.

and $$(x + y)(x - 4) = 0.$$

From which we get the equations,

and $\left.\begin{array}{r} x-y-4=0 \\ x+y=0 \end{array}\right|$ (R), $\left.\begin{array}{r} x-y-4=0 \\ x-4=0 \end{array}\right|$ (S), $\left.\begin{array}{r} x-2y+1=0 \\ x+y=0 \end{array}\right|$ (T),

$\left.\begin{array}{r} x - 2y + 1 = 0 \\ x - 4 = 0 \end{array}\right|$ (U).

From (R), we get the system of values, $x = 2$, and $y = -2$.
From (S), " " " $x = 4$, and $y = 0$.
From (T), " " " $x = -\frac{1}{3}$, and $y = +\frac{1}{3}$.
From (U), " " " $x = 4$, and $y = \frac{5}{2}$.

Take the equations, $x^2 + y^2 - 2yx - 4y + 4x = 0$, (A),

and $$x^2y^2 - y^2 - 6xy + 5 + 4y = 0, \text{ (B)}.$$

From (A), we get $(x-y)^2 + 4(x-y) = 0$,

or, $$(x-y)(x-y+4) = 0, \text{ (A}'\text{)}.$$

And from (B), we get (by adding and subtracting 4),

$$x^2y^2 - 6xy + 9 + 4y - y^2 - 4 = 0,$$

or, $(xy-3)^2 - (y-2)^2 = 0$, or, $(xy+y-5)(xy-y-1) = 0$, (B′),

(A′) and (B′) give the system of equations,

and $\left.\begin{matrix} x-y=0 \\ xy+y-5=0 \end{matrix}\right|$ (G), $\left.\begin{matrix} x-y=0 \\ xy-y-1=0 \end{matrix}\right|$ (H), $\left.\begin{matrix} x-y+4=0 \\ xy+y-5=0 \end{matrix}\right|$ (I), $\left.\begin{matrix} x-y+4=0 \\ xy-y-1=0 \end{matrix}\right|$ (K).

From (G), we get the values,

$$\left| \begin{matrix} x' = -\frac{1}{2} + \frac{\sqrt{21}}{2}, \ x'' = \frac{-1-\sqrt{21}}{2}, \\ y' = \frac{-1+\sqrt{21}}{2}, \ y'' = \frac{-1-\sqrt{21}}{2}, \end{matrix} \right.$$

From (H), we get the values,

$$\left| \begin{matrix} x' = \frac{1+\sqrt{5}}{2}, \ x'' = \frac{1-\sqrt{5}}{2}, \\ y' = \frac{1+\sqrt{5}}{2}, \ y'' = \frac{1-\sqrt{5}}{2}. \end{matrix} \right.$$

From (K), we get the values,

$$\left| \begin{matrix} x' = \frac{-5+\sqrt{29}}{2}, \ x'' = \frac{-5-\sqrt{29}}{2}, \\ y' = +\frac{3+\sqrt{29}}{2}, \ y'' = +\frac{3-\sqrt{29}}{2}. \end{matrix} \right.$$

517. One of the equations only may be capable of decomposition into factors.

The equations may be of the form, $A = 0$, and $BD = 0$. We will then have two systems of values; one resulting from $A = 0$ and $B = 0$, the other from $A = 0$ and $D = 0$.

Take the equations,

$$x^2y^2 - 7xy - 20x = 0, \text{ (A), and } (xy - 5x + 3)(x - 2) = 0, \text{ (BD).}$$

We get the system of equations,

$$x^2y^2 - 7xy - 20x = 0, \text{ (A), and } xy - 5x + 3 = 0, \text{ (B).}$$

From which, $x' = 3$, and $x'' = \frac{2}{5}$, $y' = 4$, and $y'' = -\frac{5}{2}$.

(A) and (D) give $x^2y^2 - 7xy - 20x = 0$, and $x - 2 = 0$.

From which we get $x = 2$, $y' = \dfrac{7 + \sqrt{209}}{4}$, $y'' = \dfrac{7 - \sqrt{209}}{4}$.

518. If any of the successive remainders be capable of decomposition into factors, which are functions of x or y, these factors may be placed, separately, equal to zero, and the deduced values of x or y substituted in the preceding divisor.

For, let R be one of the dividends, R′ the divisor, Q the quotient, and $f(x) \times f(y)$ the remainder.

Then, $\dfrac{R}{R'} = Q + \dfrac{f(x) \times f(y)}{R'}$, or $R = R'Q + f(x) \times f(y)$.

And this equation will be satisfied when $R' = 0$ and $f(x) = 0$, or when $R' = 0$ and $f(y) = 0$.

Take the equations, $yx^3 + y^2x^2 - x^2 - y^2x + yx + y^2 - 1 = 0$, (R).

and $x^2 - y + 1 = 0$, (R′).

Dividing R by R′, we will get a remainder, $(x^2 + 1)(y^2 - 1)$

Placing $x^2 + 1 = 0$, we get $x = \pm\sqrt{-1}$, and this, substituted in (R), gives $y = 0$. Placing $y^2 - 1 = 0$, we get $y = \pm 1$, and these values, when substituted, give $x' = 0$, and $x'' = \pm\sqrt{-2}$.

Hence, we have the system of values,

$x' = +\sqrt{-1}$	$x'' = -\sqrt{-1}$	$x''' = 0$	$x^{iv} = \sqrt{-2}$	$x^{v} = -\sqrt{-2}$
$y' = 0$	$y'' = 0$	$y''' = 1$	$y^{iv} = -1$	$y^{v} = -1$

All of which will satisfy the given equations.

519. When it is necessary to multiply (A) by either a function of x or y to make it divisible by (B), we can tell whether the multiplier has introduced foreign values, by combining it, placed equal to zero, with $B = 0$. If any of the values thus found are the same as those resulting from the combination of the given equations, we must reject these common values from the solutions of the given equations.

For, let $A = 0$, and $B = 0$; and suppose that (A) will not be divisible by B until it has been multiplied by $f(x)$. Then we will have $A(fx) = 0$, and $B = 0$, which can be satisfied when $A=0$, and $B = 0$; or, when $f(x) = 0$ and $B = 0$.

Now, it is plain that, if the combination of the new equation, $Af(x) = 0$, with $B = 0$, gives, among its system of values, the same as given by $f(x) = 0$, and $B = 0$, we must reject the common values as having been introduced by the multiplication of $f(x)$.

Take the equations, $x^2y - 2xy + x^2 = 0$, (A), and $x^3y - 2x^2 + x = 0$, (B).

Multiplying (A) by x^3, to prepare for division, we get for the final equation, $x^2(x^3 + 2x^2 - 5x + 2) = 0$. From which we get, $x = 0$, $x' = 1$, $x'' = \frac{-3 + \sqrt{17}}{2}$, $x''' = \frac{-3 - \sqrt{17}}{2}$. The corresponding values of y are, $y' = \frac{0}{0}$, $y'' = \frac{6\sqrt{17} - 26}{38 - 10\sqrt{17}}$, $y''' = \frac{-(26 + 6\sqrt{17})}{38 + 10\sqrt{17}}$ (M).

Now, placing $f(x) = 0$, and $B = 0$; or, $x^2 = 0$, and $x^3 - 2x^2 + x = 0$, we get $x = 0$, and $y = \frac{0}{0}$; and, since these values are the same as the first of those, marked (M), we must reject them from (M), and leave but three values for x, and three for y.

520. If A be exactly divisible by B, the values of x and y will be indeterminate.

For, then, $\frac{A}{B} = Q$; and, since $A = 0$, and $B = 0$, we will have $\frac{0}{0} = Q$, or $0 = 0Q$. (P). Now, it is plain that Q may be $f(x)$, or $f(y)$, or $f(x, y)$. But, equation (P) will be satisfied, whatever may be the form of Q, and whatever may be the values of x or y.

Take the equations,

$$x^4 - 3x^3y - 2x^3 + 3y^2x^2 + 6x^2y - y^3x - 6y^2x + 2y^3 = 0,$$

and

$$x^2 - 2xy + y^2 = 0.$$

We get as a quotient, $x^2 - xy - 2x + 2y$, and a remainder zero, and any value whatever of x, with the corresponding or deduced value of y, will satisfy both equations.

Remark.

The case exhibited in 520, differs only from that in 514 in this respect, there is a common factor to the two equations in both instances, but 520 does not manifest that factor.

The indeterminate nature of the given equations, when the final equation in y is zero in both members, may also be shown by retaining the trace of y. For, when we place the zero remainder equal to zero, we have $0y = 0$, or $y = \frac{0}{0}$.

521. When the final equation reduces to a constant, the value of y will be infinite, and the given equations will be contradictory.

For, then we will have $0y = A$, constant; or $y = \infty$, the symbol of absurdity.

Hence, the combination of the given equations has led to an absurdity, and, therefore, these equations must be contradictory.

Take the equations, $y^3 - x^3 - 3x^2 + 9 - 3x = 0$,

and $$y - x - 1 = 0.$$

Combining, we will have, for the final equation, $10 = 0$; or, retaining the trace of y, $10 + 0y = 0$, or $y = -\frac{10}{0} = \infty$.

The given equations are not simultaneous.

GENERAL EXAMPLES.

1. $\left| \begin{array}{l} y^2 - 2xy - 4x + x^2 = 0, \\ y + x - 4 = 0. \end{array} \right.$

Ans. $x' = 4$, $x'' = 1$; $y' = 0$, $y'' = 3$.

2. $\left| \begin{array}{l} y^2 - 2xy - 4x - x^2 = 0, \\ y^2 - 2xy - 5x - 2x^2 + 2 = 0. \end{array} \right.$

Ans. $x = 1$, or -2, $y = 1 \pm \sqrt{5}$, or $-2 \pm \sqrt{-4}$.

3. $\left| \begin{array}{l} yx^5 - 3y^2x^4 + 3y^3x^3 - 4y^4x^2 + 5y^5x - 2y^6 = 0 \\ x^2 - 3yx + 2y^2 = 0. \end{array} \right.$

Ans. x and y indeterminate.

4. $\left| \begin{array}{l} x^6 - 2yx^4 + x^3 + x^2 - 2yx + y^2 = 0, \\ x^2 - 2yx + 1 = 0. \end{array} \right.$

Ans. $x = \pm 1$, or $\pm\sqrt{-3}$, $y = \pm 1$.

5. $\left| \begin{array}{l} x^5 - 2yx^4 + x^3 + x^2 - 2yx + 1 = 0, \\ x^2 - 2yx + 1 = 0. \end{array} \right.$

Ans. $x = \frac{0}{0}$, $y = \frac{0}{0}$.

6. $\left| \begin{array}{l} y^2 - 2xy + x^2 - 2y - 1 = 0, \\ y^2 - 2xy + x^2 + x = 0. \end{array} \right.$

Ans. $x = -1$, or $-\frac{1}{9}$; $y = 0$, or $-\frac{4}{9}$.

7. $\left| \begin{array}{l} y^2 - 2xy + 2x^2 - 2y + 4 = 0, \\ y^3 - 4xy + 7x^2 - 2y^2 - 5 = 0. \end{array} \right.$

Ans. Values imaginary.

8. $\left| \begin{array}{l} y^2 - 4xy + 5x^2 + 2x + 1 = 0, \\ y - 2x = 0. \end{array} \right.$

Ans. $x = -1$, $y = -2$.

9. $\left| \begin{array}{l} y^2 + xy - 2x^2 + 3x - 1 = 0, \\ y^2 - x^2 = 0. \end{array} \right.$

Ans. $x = \frac{1}{3}$, or 1, or $\frac{1}{2}$, and $y = \frac{1}{3}$, or -1, or $-\frac{1}{2}$.

10. $\left| \begin{array}{l} (yx - 1)(x - 2)(y - 4) = 0, \\ (x + 2)(yx - x)(x - 3) = 0. \end{array} \right.$

Ans.	Values of x.	Values of y.
	$x = -2$	$y = -\frac{1}{2}$
	$x = 1$	$y = +1$
	$x = 3$	$y = +\frac{1}{3}$
	$x = 0$	y incompatible
	$x = 2$	$y = 1$
	x incompatible	y incompatible
	$x = -2$	$y = 4$
	$x = 0$	$y = 4$
	$x = 3$	$y = 4$

11. $\left| \begin{array}{l} y^2 - 2xy + x^2 + 2y = 0, \\ y^2 - 2xy + x^2 - 2y - 1 = 0. \end{array} \right.$

Ans. $y = -\frac{1}{4}$, $x = -\frac{1}{4} \pm \sqrt{\frac{1}{2}}$.

12. $\left| \begin{array}{l} y^2x^4 - yx^4 - 8y^2x - 2yx^3 + 8yx + 16y + 2x^3 - 16 = 0, \\ yx + y - x - 1 = 0. \end{array} \right.$

Ans. $y = 1$, or -2; $x = \frac{0}{0}$, or -1.

Final equation, $y^2 + y - 2 = 0$.

13. $\left| \begin{array}{ll} y^3 - x^3 + 7 = 0, & \text{(A)} \\ y^2 - x^2 + 3 = 0. & \text{(B)} \end{array} \right.$

Ans. $x = 2$, and $y = 1$.

After the first division, multiply (B) by $(x^2 - 3)^2$, we will get for the final equation, $9x^4 - 14x^3 - 27x^2 + 76 = 0$. The only rational value in this equation is $x = 2$.

14. $\left| \begin{array}{l} y^4 - x^4 - 15 = 0, \\ y^3 - x^3 - 7 = 0. \end{array} \right.$

Ans. $x = 1$, and $y = 2$.

Final equation, $(x^4 + 15)^3 - (x^3 + 7)^4 = 0$.

Required the values and the final equation belonging to

15. $\left| \begin{array}{l} y^5 - x^5 - 2101 = 0, \\ y^4 - x^4 - 369 = 0. \end{array} \right.$

THE END.